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ABSTRACT

This is part one of a two-part SMSG textbook for inservice education of elementary teachers. Part one contains Unit A and 5 chapters of Unit B. Chapter titles of Unit A are: The Mathematics Programs, Grades K-3; Description of Culturally Disadvantaged Children; Language and Mathematical Instruction; and Number Concepts of Disadvantaged Children. The Unit B chapters are: Sets; Comparing Sets; Whole Numbers; Set Operations; and Elements of Geometry. (MK)

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# INSERVICE COURSE FOR ELEMENTARY SCHOOL TEACHERS

**SPECIAL EDITION (Preliminary)**

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# INSERVICE COURSE
# FOR ELEMENTARY
# SCHOOL TEACHERS

## SPECIAL EDITION (Preliminary)

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# Preface

A sound knowledge of mathematics is becoming a prerequisite for fruitful work in an ever-increasing number of endeavors. This knowledge must include why mathematical processes work as well as how they work. It is not enough for today's children to learn mathematics by rote.

Children now in elementary school can be expected to face problems which now we cannot foresee. These problems will be solved not by knowledge of mathematical facts alone, but by knowledge of mathematical methods of attacking problems. New and as yet unknown problems may involve, and in fact will require, new and as yet unknown mathematics for their solutions. Naturally, we cannot teach this unknown mathematics, but we can and must teach methods of mathematical thinking as well as the basic content of mathematics.

In general, schools today are becoming increasingly aware of the need to orient the teaching of mathematics in accord with a conceptual development of mathematical ideas. Yet, too frequently an assumption is made that for a population of children that is considered to be "culturally deprived", rote learning is still the only answer to learning mathematics. This course of action would further deprive these children. A feature of this text is an attempt to attend to problems that may be associated with the culturally disadvantaged. Unit A of this book contains four chapters of general background information and Unit B consists of fourteen chapters dealing with mathematical content.

In Unit A, Chapter 1 is an overview of the course. Chapter 2 begins with a description of the culturally disadvantaged based on psychological findings. It continues with the physical, social and psychological environment in which these children function in their pre-school years. The next concern is with the characteristics manifest in the culturally disadvantaged children as they enter school. Finally, implications for teaching these children are discussed in this chapter.

Chapter 3 attends to language and mathematical learning. The careful building of understanding and correct use of mathematical language through aural-oral experiences is considered. Particular examples and suggestions useful to teachers of young children are included.

Chapter 4, on general background, contains information gathered from observations and testing of children who used the School Mathematics Study Group texts, MATHEMATICS FOR THE ELEMENTARY SCHOOL, Books K and 1 during the school year 1964-65. These results are taken from a comparative study of children described in Chapter 2.

The remainder of the book presents mathematical content relevant to teaching in the primary grades. All topics which are included in the texts mentioned above are treated, but from a more sophisticated point of view. Other topics have been included when the development has warranted it. As the student texts, Books K and 1, do not develop topics in the same sequence, adaptation of this course to a particular grade may necessitate some minor change in the order of studying these materials. The book has been written with an inservice course in mind; however, it is hoped that the text is sufficiently lucid to be understood easily by the reader.

It is assumed that greater comprehension and interest will derive from discussions between an instructor and inservice teachers. For this purpose, the problems that have been inserted at appropriate intervals in each chapter (as opposed to a set of exercises at the end of each chapter) may be an integral part of discussions for clarifying fine points and for deepening understanding.

When it was felt that some comment on pedagogy or other relevant remarks might contribute toward better understanding and teaching of the concepts, these comments are included under the section, Applications to Teaching, at the end of the chapter. It is important to note that a large part of what is presented here is background material for you, as a teacher, and is not intended to be transmitted to your student per se. We hope that as you read the text and do the exercises, you will increase your understanding of some basic notions underlying the mathematics that you are teaching, and that, in general, this text will prove useful.

In the preparation of this book, we have drawn on materials produced by the School Mathematics Study Group, and in particular, STUDIES IN MATHEMATICS, Volume IX, A BRIEF COURSE IN MATHEMATICS FOR ELEMENTARY SCHOOL TEACHERS. For the use of these materials, we offer acknowledgement.

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# UNIT A

## Chapter 1

## THE MATHEMATICS PROGRAM, GRADES K-3

The SMSG mathematics program, MATHEMATICS FOR THE ELEMENTARY SCHOOL, K-3, is a contemporary instructional program that emphasizes conceptual learning. Primary attention is given to the introduction and progressive development of significant mathematical ideas. This emphasis on mathematical ideas provides the necessary foundation for the related development of appropriate skills and the ability to use mathematics effectively.

Central to the program are relatively few basic ideas. Two of these are the ideas of number and operation. Each is introduced and extended in close association with appropriate manipulations of sets of physical objects. Major attention is given to the set of whole numbers, $\{0, 1, 2, 3, 4, \ldots\}$, and to the nature and properties of the familiar operations of addition, subtraction, multiplication, and division within the set of whole numbers. Consideration is given also to the non-negative rational numbers, such as $\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \frac{5}{6}, \frac{7}{8}$.

Closely related to the ideas of number and operation is the idea of numeration and also the ability to compute. Emphasis is given to the decimal base (ten) of our numeration system and to its "place value" principle. These, coupled with properties of the operations, form the basis for developing meaningful algorithms,--i.e. forms for computing.

The remaining major idea developed in the program is geometric in nature. Our first concern is with characteristics and properties of familiar geometric figures, viewed as sets of points and abstracted from appropriate models within the physical world. A related concern is with measurement, in which number is applied to properties such as length and area of geometric figures.

The Teachers' Commentary for Book K stresses informal work with pre-number concepts at the kindergarten level. These are concepts pertaining to sets, and are explored through work with sets of physical objects. The ideas of number and geometry also are introduced at the kindergarten level, as reflected in the following table of contents.

Book K.

Overview

Mathematical Background

Chapter

1. SETS

    Activities to Help Develop These Concepts
        Sets
        Members of a set
        Sets with a single member
        Sets with no members: the empty set

    Additional Activities

2. RECOGNIZING GEOMETRIC FIGURES

    Activities to Help Develop These Concepts
        Circle, rectangle, triangle, and square
        Use of regions
        Curves
        Use of solids
        Use of regions and solids
        Inside, outside and on

    Additional Activities

3. COMPARISON OF SETS

    Activities to Help Develop These Concepts
        As many as
        More than
        Fewer than

    Additional Activities

4. SUBSET OF A SET

    Activities to Help Develop This Concept

5. JOINING AND REMOVING
    Activities to Help Develop These Concepts
        Joining
        Removing subsets and the remaining set
6. COMPARISON OF SIZES AND SHAPES
    Activities to Help Develop These Concepts
        Large and small
        Long and short
        Narrow and wide
        Short and tall
    Additional Activities
7. ORDERING OF SETS
    Activities to Help Develop This Concept
8. ADDITIONAL ACTIVITIES TO DEVELOP CERTAIN MATHEMATICAL IDEAS
    Activities to Help Achieve Desired Results
        Using geometric shapes
        Using "position" words
9. USING NUMBERS WITH SETS
    Activities to Help Develop These Concepts
        Number of members in selected sets
        Pairing with an imaginary set
        Sets containing two members (pairs)
        Sets equivalent to a model set
        Problem solving
    Additional Activities
Materials
Miscellaneous Ideas
Bibliography
Kindergarten Vocabulary

The program of Book 1 reviews and extends pre-number concepts associated with sets, and also reviews and extends the work with numbers and geometry. Operations and numeration are introduced, along with measurement. The extent of this work is reflected in the following table of contents.

# Book 1.

Chapter

I. SETS AND NUMBERS

1. Set and member
2. The empty set
3. Pairing and equivalence
4. Comparison of sets
5. Review
6. Numbers and equivalence
7. Number perception, without counting
8. Comparison of numbers

II. NUMERALS AND THE NUMBER LINE

1. Arranging sets in succession
2. The numerals 0 through 9
3. The number line

III. SETS OF TEN

1. Sets of ten
2. Naming multiples of ten
3. Application: money
4. Problem solving with sets of ten

IV. INTRODUCTION TO ADDITION AND SUBTRACTION

1. Joining
2. Joining sets and counting
3. Joining sets and adding numbers
4. Subsets
5. Removing sets and the remaining set
6. Removing sets and subtracting numbers

V. RECOGNIZING GEOMETRIC FIGURES

1. Familiar three-dimensional shapes
2. Simple closed curves
3. Polygons
4. Classifying regions
5. Fitting regions

## VI. PLACE VALUE AND NUMERATION

1. Counting by tens and ones
2. Eleven, twelve, and the teens
3. Spoken names of the numbers: 21 through 99
4. The written numerals: 20 through 99
5. Order relations for numbers 0 through 99

## VII. ADDITION AND SUBTRACTION

1. Partitions and addition
2. Partitions and subtraction
3. Addition and subtraction on the number line
4. How many more
5. Problem solving and equations
6. Addition and subtraction: numbers greater than ten

## VIII. ARRAYS AND MULTIPLICATION

1. Arrays
2. Multiplication
3. Simple properties of multiplication

## IX. PARTITIONS AND RATIONAL NUMBERS

1. Partitioning sets into equivalent subsets
2. Partitioning into a given number of equivalent sets
3. Parts of regions and sets
4. One-half
5. One-third
6. Using rational numbers

## X. LINEAR MEASUREMENT

1. Line segment, straightedge
2. Comparing line segments
3. Measurement of line segments
4. Construction of a ruler

It is imperative that kindergarten and first-grade teachers view their instructional work in relation to the work of subsequent grades. All of the basic ideas of the program appear within Book 1 and are extended in Book 2 and Book 3 as reflected in the following tables of contents.

## Book 2.

Chapter

I. SETS AND NUMBERS: REVIEW

1. Sets
2. Comparing Sets
3. Joining Sets and Adding Numbers
4. Removing Sets and Subtracting Numbers
5. Doing and Undoing
6. Partitioning Sets into Equivalent Subsets
7. Place Value
8. The Number Line
9. Comparing Numbers
10. Even and Odd Numbers

II. ADDITION AND SUBTRACTION: REVIEW

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2. Partitions
3. The Addition Table
4. Subtraction
5. Missing Addends
6. Problem Solving
7. Comparing Numbers: Review and Extension
8. Using the Associative and Commutative Properties
9. Developing Problem Solving Skills

III. SETS OF POINTS

1. Points
2. Curves, Line Segments
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5. Triangles

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2. Simple Addition and Subtraction
3. Addition and Subtraction Facts for Sums of 11 through 14
4. Addition and Subtraction Facts for Sums of 15 through 18
5. Problem Solving

V. LINEAR MEASUREMENT

1. Comparing Sizes
2. Congruence of Line Segments
3. Units of Length
4. Measure and Length of Line Segments
5. Length to the Nearest Unit
6. Using a Standard Unit of Length
7. Linear Scale

VI. COMPUTING SUMS AND DIFFERENCES

1. Place Value
2. Adding and Subtracting: Tens and Ones
3. A Vertical Algorithm for Addition
4. Computing Sums Involving Renaming Ones as Tens
5. Column Computation
6. Subtraction, Renaming Tens as Ones
7. Using "n" in Equations and Problems
8. Monetary Units and Place Value

VII. CONGRUENCE OF ANGLES AND TRIANGLES

1. Congruence of Triangular Regions
2. Rays
3. Angles
4. Congruence of Angles
5. Right Angles

VIII. ARRAYS AND MULTIPLICATION

1. Arrays
2. Multiplication
3. Some Simple Properties of Multiplication
4. The Distributive Property and Multiplication Facts

IX. DIVISION AND RATIONAL NUMBERS

1. Finding Factors
2. Division, Using Arrays
3. Division, Using Subtraction
4. Rational Numbers
5. Rational Numbers and the Number Line

## Book 3.

Chapter

I. SETS OF POINTS

1. Points, Curves and Line Segments
2. Lines, Rays and Angles
3. Simple Closed Curves, Polygons
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5. Some Special Triangles

II ADDITION AND SUBTRACTION: REVIEW AND EXTENSION

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2. Comparison of Sets; Order Among Numbers
3. Addition and Subtraction Facts
4. Place Value
5. Techniques for Finding Sums
6. Techniques for Finding Differences
7. Problem Solving
8. Extensions

III. DESCRIBING POINTS AS NUMBERS

1. Coordinates of Points on a Line
2. Motions on the Number Line
3. Coordinates in a Plane
4. Plane Figures
5. Stretching Pictures of Segments on a Line
6. Enlarging Pictures
7. Scale Drawing

IV. ARRAYS AND MULTIPLICATION

1. Arrays
2. Multiplication
3. The Basic Multiplication Facts
4. Prime Numbers

V. ADDITION AND SUBTRACTION: SHORTER FORMS OF COMPUTATION

1. Adding and Subtracting Multiples of Ten and One Hundred
2. A "Short Form", Renaming Once
3. Finding the Sum of Three Numbers
4. Adding, with Renaming More than Once
5. Subtracting, with Renaming More than Once

VI. LENGTH AND AREA

1. Length of a Curve
2. Perimeters of Polygons
3. Using Different Units
4. Using Fractional Units
5. Introduction to Area
6. Doubling Edges of Rectangles
7. Congruence and Area
8. Gauss Sums and Euler Routes

VII. MULTIPLICATION, QUOTIENTS, AND DIVISION

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2. Using Multiplication to Solve Problems
3. Division
4. Quotients
5. Relating Division to Multiplication
6. The Associative Property of Multiplication
7. Multiplying and Dividing 10 and Multiples of 10
8. The Distributive Property
9. Solving Problems
10. Factors and Clock Wheels (Optional)

## VIII. RATIONAL NUMBERS

1. Rational Numbers Associated with Parts of Regions
2. Rational Numbers Associated with Subsets of a Set
3. Identifying a Subset as a Part of a Set
4. Finding a Part of a Set
5. Rational Numbers Describe Points of the Number Line
6. Order among Rational Numbers

## IX. DIVISION

1. Division Concepts
2. Finding Quotients
3. Finding a Number that Names Parts of a Set
4. A Technique for Finding Quotients
5. Solving Problems

The latter part of this book develops the mathematics underlying the program of Grades K-3. By using the chapter outlines above, the teacher will be able to see how these ideas are progressively developed from kindergarten through Grade 3.

The nature and scope of mathematical content embraced by MATHEMATICS FOR THE ELEMENTARY SCHOOL, K-3 interest and challenge children within the primary grades. They also interest and challenge the teachers of these children. Through this program mathematics truly can "come alive" for both children and teachers. May it be that for you and the children you guide in their mathematical explorations and development.

# Chapter 2

# DESCRIPTION OF CULTURALLY DISADVANTAGED CHILDREN

## I. Introduction

From a variety of sources, data have been accumulating which explain the disadvantaged position of the culturally deprived child as he starts school. If we review some of the conditions within the family and neighborhood which these children experience during their pre-school years, then the characteristics of these disadvantaged children become more meaningful.

The two major criteria used in defining people classified as culturally disadvantaged are (1) low economic status, and (2) lack of participation in middle-class culture. The actual family income may vary from one study to another. A maximum family income of $2,000 per year defines the disadvantaged in some studies; an income below $4,000 may define this group for others working with them.

The criterion of lack of participation in middle-class culture is more difficult to specify, but relates most closely to the values placed upon education. The lack of books, of parental examples of reading and success in education, and the lack of stimulation to achieve are all parts of this non-participation in middle-class culture.

The culturally disadvantaged group consists mainly of urban slum-dwelling people, particularly because the United States population is becoming increasingly more urban. This fact does not, however, preclude inclusion of such marginal subsistence groups as segregated rural Negroes, dwellers in the depressed areas of Appalachia, and many American Indian groups from the ranks of those described as culturally disadvantaged.

## II. Contributing Factors

If we look to the home and environmental circumstances that influence these children in ways which are apparent at school entry, the physical living conditions as well as the quality of the parent-child interactions are most striking.

A. Physical Living Conditions

The living conditions provide a particular kind of setting within which the parent-child interactions take place. The crowding of dwellings in disadvantaged areas of large cities allows for little privacy, solitude, or quiet. Not only are there likely to be many people occupying a small apartment, but the dwelling units are close upon each other. In other words, the density of people for the physical space is very high. What this means for a young child is that he has almost no place to play without being either "underfoot" or out on the street. He is constantly subjected to noise from the family, television, neighboring households, and street activity.

The child in the disadvantaged home is not likely to have books or magazines available to look at nor to have read to him. He is not likely to have a variety of toys with which he can amuse himself nor toys which encourage sharing. The possibility of developing gross and fine muscle coordination and independent imaginative play through drawing, cutting, and building blocks, for example, are lacking. He is less likely to have been taken on trips outside the immediate neighborhood--to the zoo, parks, a farm, museums, or even to the library. Thus, the experiences of these children prior to school entry have been different from those of children of middle-class families and much more highly restricted in variety.

B. Hostility of the Environment

In the above section the typical household situation was described. The character of the neighborhood, as the broader social setting, also influences these children in ways which are apparent at school entry.

The environment of the disadvantaged is described as hostile because of the higher rates of delinquency, disease, and death in their neighborhoods.[1] Whether or not these conditions can be called hostile, the following conditions are, at the least, not conducive to moving outside the home or relating to the community. First, fewer public recreational facilities are located in these areas than in areas of higher income residences. Second, school buildings in these areas tend to have less adequate equipment and facilities for such activities as science and art, to mention two of many which could be cited. Third, contacts outside the immediate neighborhood have frequently been with authority figures-policemen, welfare workers, or school officials. The school is likely to be associated with this authority and feared rather than being considered an important resource for help and development of potential.

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[1] For studies supporting these and the immediately following statements, see Sexton, Patricia C. Education and Income. New York: The Viking Press, 1961.

C. Parent-Child Relationships

Since the basic family unit among many of the disadvantaged groups does not consist of mother, father and their children, the effects of other compositions of the family unit must be considered. In many of these homes the father is not present. The household often consists of mother, children, and possibly other female adults such as an aunt or maternal grandmother. There may be considerable instability both in the living arrangements and in the adults important to the child. For example, a maternal aunt and her children may move into the household if there is a crisis in their lives; or a child may leave his mother and move to a relative's home if his mother takes a job.

What emerges is a form of "extended family" which provides a certain safety and security against what may be perceived as the hostile world. What lack appears most significant, especially for the child's school performance, is that of a directed interaction between the adult and the child. The mother in the culturally disadvantaged home is not likely to spend time in conversing with one of her children alone, nor in sitting down to teach him a specific skill such as tying his shoelaces. Supervision of the child is handled by any of the adults available or frequently by older siblings.

To clarify the term "adult-child interaction", let us use two typical situations in a two-year-old's exploration of his environment. He may reach for something hot, or poke a finger into the eye of his baby sister. Mother, who is likely to be preoccupied with the sheer physical demands of life, does not explain why the child's behavior will be harmful to him or the baby; rather she will yell at him, "Stop that!", or "Bad boy!", or simply slap him.

The implications of mother's response for the two-year-old are several. First, there is no verbal specification of what, exactly, the undesired act was. He may interpret mother's slap as meaning that reaching or touching is wrong and, therefore, punishable. Thus, there is no opportunity for learning a discrimination between the act of reaching or touching and the consequences of reaching for certain objects (in this instance, hot) or touching (the baby's eye). What this kind of punishment is likely to achieve, if used as the usual means of discipline, is a stifling of reaching and touching. This will eventually diminish the child's curiosity by reducing his explorations of his immediate environment.

A second implication of mother's use of "Stop that!" or a slap is that it does not provide a model for complex verbal behavior. The child needs to listen to language forms in order to pattern his own language, both in terms of range of vocabulary and complexity of expression. Also, he needs the experience of verbally expressing his questions, reasons, and feelings in order to learn to communicate verbally.

A third implication of mother's response in the earlier cited examples is for the child's self-concept. If mother had said in response to his reaching or poking, "What a big boy you are to be able to reach so high (onto the stove or into the crib)! But you must be careful about hot things or baby's eyes," then there is some increment in a positive concept of himself. He is growing and is capable of new accomplishments. By slapping or telling him that he is a bad boy, however, his image of himself is deflated. What is likely to evolve in this setting is an image of the good child as one who stays out of the way and who is quiet. This is not the child who will achieve in school through high motivation and striving.

A further point should be made about the relationships between mother and child as contributing to the child's behavior as we see him at school entrance. If the mother has not had many years of formal education herself, she will be less aware of what experiences she could provide which would eventually help her child in school.

### D. Planning and Scheduling of Time

A characteristic of many disadvantaged families, partly related to their living conditions and partly related to their sub-culture, is the lack of a family schedule or routine. Meals are not eaten at regular times, nor is there a set bedtime for the children. It is seldom that the family sits down to a meal together.

There are two effects of this lack of time-planning and routine which are likely to cause difficulties in the young child's adjustment to school. The first has to do with adapting to a routine and working independently within time limits. The second has to do with verbal development.

Let us deal, first, with the use of time. The child who has not experienced some scheduling of activities at home will not be able, without considerable help, to adapt to a school routine - a time to start a given activity with the class and a time to finish up that activity when directed. This means less self-direction and less ability to work independently. In addition, being on time has no meaning unless expectations have been established that certain events occur at particular times and some consequences may follow from not being on time.

Without the experience of planning time and using periods of time within the day for particular activities, the child is less likely to be able to develop longer-range, more abstract goals which involve both planning longer blocks of time and sequencing time. It may well be that the demands of career plans involving particular steps both in the immediate and more remote future are not possible without these early experiences. Successful performance in specially selected courses in high school, along with summer jobs to earn money, in order to enter college involves such sequenced, long-term planning. When a mother explains to her pre-school child that he may play with his friend at a certain time, after his nap, or when she says that he may watch television until supper at 5:30, she may be laying the foundation for later longer-range, goal-oriented planning.

The fact that the family does not sit down together for a meal or for discussing the day's happenings permits fewer opportunities for verbal interaction. The child does not have the experience of hearing, attending to, or participating in complex verbal expression. The child who does not participate with adults in such verbal interchange has little opportunity to be heard, to be corrected, and therefore to have his language modified and expanded. At school entrance, the child comes into a situation where there are expectations that he express his ideas in verbal interchange with an adult. The situation is strange and unfamiliar. He must learn to adapt to this new kind of interaction as well as to learn the language necessary in order to participate.

In discussing the mother-child interactions earlier in this chapter, it was pointed out that the mother does not use complex verbal explanations in directing her child's behavior. More physical, rather than verbal, means of discipline, plus the lack of conversation among family members, combined with the lack of direct teaching, contribute to the development of a child experientially limited in both the content (vocabulary, variety, and complexity of speech forms) and the structure of such verbal communication.

### E. Lack of Preschool Experience

A source of enrichment for some young children, though usually not for the deprived child, has been a year or more of nursery school prior to school entry. Since nursery schools have traditionally been privately funded, therefore requiring tuition, they have not, in the past, been available to the disadvantaged groups. With the increasing governmental concern for the economically deprived segments of our population, such programs as Project Headstart will undoubtedly have influence on the experiential development of these children.

## III. Characteristics of Culturally Disadvantaged Children

In the two preceding sections we have described certain characteristics of culturally disadvantaged families which influence their children's behavior by the time they start school. In this section we shall describe feelings and behavior of these children in the beginning school years resulting from the family and broader environmental influences.

### A. Self Concept

Given the conditions of a hostile environment, of their families being the "have-nots" economically and socially, and of the lack of experiences directly relevant to classroom learning, these children are not likely to have positive feelings about themselves. Most crucial in the context of this book are their feelings about their competence for succeeding in school. For the Negro child especially, the effects of prejudice, segregation, and inferior status are likely to lead to negative feelings of his own worth.[2] These feelings, in turn, lead to little motivation or *striving* for success since these children are learning that their chances for success are relatively limited.

Since these children do not have the basic skills or know-how for immediate adaptation and successful performance in the classroom situation, they are likely to meet with frustration and confusion, if not failure, very quickly. The effects of such experiences will further detract from their feelings of competence.

Somewhat less tangible, but worthy of mention, is the observation that these children do not have as differentiated a self-concept as do more privileged children. By differentiation is meant the perception of one's self as a unique individual with certain characteristics, preferences, and wishes which form an identity distinguishing one from others. The reasons suggested for this lack of clearer differentiation are negative. That is, there is not an intense relationship between a parent and an individual child in these families, nor is the treatment of a given child individualized. As a result, these children display less self-concern, less competitiveness, and less motivation for self-improvement. These are facets of intrinsic motivation which many teachers rely on to keep children involved in a given task.

There is a special problem in the development of a self-concept in boys from culturally disadvantaged homes. This arises from the lack of a stable father. As pointed out earlier, in many of these homes the father is absent. There may be adult males in and out of the home, but the presence of these potential models is likely to be temporary. This situation does not allow for a stable relationship with an adult male whom the boy may use as a model for imitation and identification. The adult males with whom the child does have contact are generally not those presenting a picture of responsibility or successful achievement as measured by the standards implicit within our schools.

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[2] For a detailed discussion of this topic, see Ausubel, D. and Ausubel, Pearl, "Ego development among segregated Negro children," in Passow, A. Harry, *Education in Depressed Areas*. New York: Teachers College, Columbia University, 1963.

## B. Language

The language with which the culturally deprived child comes to school is likely to be different from the more advantaged child's in two major ways.[3] The first is in the quantity of verbal expression, and the second is in the quality of verbal expression.

Concerning quantity, children from disadvantaged homes tend to speak in short phrases. A monosyllabic response to a teacher's request or question is typical. It is not uncommon to see kindergarten children of this group sitting side by side at a table in a classroom and not having any verbal interchange at all. In certain pre-school enrichment programs it has been found necessary, in many instances, to encourage a child to express himself verbally by talking to an imaginary person over a toy telephone before he is able to speak directly with another child or with a teacher.

The difference between disadvantaged and more advantaged children is less likely to be seen in verbal labelling. That is, children from the disadvantaged groups can give names to commonly seen objects, such as a dog, pencil, box, key. If the object, or referent, is rarely experienced in low income environments,(e.g., "nest" which is experienced in rural environments or "giraffe" which requires books or a trip to the zoo), these words are not likely to be known by the children.

The major qualitative differences lie in the elaboration of languages--in complexity of grammatical expression and in the more abstract language which goes hand in hand with conceptual development. The experiences of these children have not prepared them well for simple classificatory behavior, such as comparing toys or other classroom materials on such dimensions as size, shape, color, number. They do not have the vocabulary for expressing such classifications or comparisons, nor have they had experiences which have made them attend to the attributes of objects. Children from the middle-class, both more experienced verbally and more aware of abstracting from attributes, are better able to state a concept explicitly when given pictures all of which fit that concept. On the other hand, lower-class children use more concrete attributes and not necessarily the essential ones. To illustrate this idea, John[4] studied a group of Negro lower and middle-class first graders. She presented them with pictures of four men at work: a policeman, a doctor, a farmer, a sailor. The middle-class children more frequently said they were alike because all were "men" or "people," which are category labels. The lower-class children focused on non-essential attributes with such statements as "look the same", "like each other."

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[3] This discussion will not deal with the child from a non-English speaking home since this presents another kind of language problem beyond the scope of this section.

C. Sensory development

Lacking the experiences of attending to attributes of objects as discussed in the previous section and lacking the experiences of looking at books, these children are not likely to be as ready for the discrimination and attention demanded by printed materials. Teachers working with these children have found that they are easily confused about the task to be done by many pictures or numerals on a page; they quickly lose their places; one page in a workbook is readily confused with another which has similar elements.

Whether their eye-hand coordination is less well developed is not as crucial an issue here as the fact that they have not had the opportunities to use pencils, crayons, and scissors. Their experiences in seeing printed lines and pictures within books and finding meaning from them are limited.

In the area of auditory discrimination, the disadvantaged child does not attend as well to teacher directions nor to her instruction, probably for two reasons. First, he is easily distracted by extraneous sounds or activities. Secondly, and partly resulting from the first, he is less able to discriminate what is the sound to be listened to from the noise which impinges.

D. Motivation to achieve school goals

Considering the picture presented of the pre-school environment of a child from a disadvantaged group and knowing the expectations of the school for task-oriented activity, the discontinuity between the two settings is striking. The child has much to "unlearn," as well as to learn. He must learn the expectations of the school, and especially the demands of his teacher, without considering content learning at all!

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[4] John, Vera P. "The intellectual development of slum children: some preliminary findings." Amer. J. Orthopsychiat., 1963, 33, 813-822.

Much has been written on the antagonism and defensiveness of lower-class families toward the schools and of the discrimination (often unconscious and unintentional) on the part of the school in dealing with lower-class parents and children. The worth of understanding such attitudes on both sides is not to be minimized. Quite apart from such factors, however, the difficulty in transition for the child entering school can readily be seen. He often lacks the long-range goals which can be achieved through school success, as well as the intrinsic motivation to learn for "self-improvement."

## IV. Implications for Teaching

In the previous section of this chapter, a picture was sketched of the child from a disadvantaged group at the time he entered school. In this section an attempt will be made to apply the understandings derived from the description of contributing factors and resulting characteristics to the teaching of disadvantaged children. In order to discuss teaching, references will be made to the performance of these children on school tasks and in test situations.

### A. Implications for Teacher Attitude

It is very important for a teacher in the primary grades to be aware that the performance of these disadvantaged children in their early school years is not necessarily a good indication of their potential. Their earlier experiences have not well prepared them for the demands made by the school; therefore, they are not as ready for school. It should be kept in mind that their rate of learning can be very rapid on tasks which do not depend on prior learning that they have not had.

In this same vein, test results should be interpreted within the context of your knowledge of these children. Specifically, there are a number of facets of standardized tests and of the testing situation which contribute to their poorer performance. First, we can go back to the visual and auditory discrimination limitations discussed in Section

III of this chapter. If the test directions are presented verbally, the child may not clearly understand what he is expected to do--assuming that he is able to attend sufficiently long to hear what is said. Then, assuming that the test is group administered, he is expected to use a pencil to mark certain symbols on a paper or booklet which involves both some dexterity in using the pencil and making rather fine discriminations among the symbols on the page.

A test which is timed adds another factor contributing to the poorer performance of disadvantaged children. These children are not as accustomed to working within a time schedule as many middle-class children are.

Other factors that contribute to poorer test performances of this group are particularly relevant to intelligence tests. These are lack of practice or test "know-how," lack of motivation to do well, inadequate rapport with the examiner, and the cultural loading of the tests themselves which discriminates against these children.

This discussion of the factors influencing test performance is placed under the heading, "Implications for Teacher Attitude," for the purpose of increasing your awareness that a set of test scores does not permit accurate judgments of what the capacities, learning rates, or potential performance of disadvantaged children may be in the future. Such test scores give information on how a given child is performing at a particular point in time. How the same child might perform given opportunities to compensate for some of these limiting conditions is a major challenge to the schools at present, in attitude, as well as in behavior.

The final admonishment concerning attitudes toward the culturally disadvantaged child is to keep in mind his earlier experiences which may make for difficulties in his relationship to you, as his teacher, in his adaptation to the school routine, and in his unfamiliarity with the work expected.

B. Implications for Teacher Behavior

Given some knowledge of the background and resultant characteristics of disadvantaged children, what can a teacher do to aid in their development and school progress? Four suggestions are given here which, it is hoped, will provide guidelines for your work and relationship to them.

The first suggestion, and perhaps the most important, is to maintain a warm and supportive relationship with these children. Although this may sound like an oft-stated platitude, it is particularly important for the groups that we are dealing with, and it can be acted upon in a variety of meaningful ways. In as many ways as possible, provide experiences which will enable the children to be successful. Conversely, avoid situations that may produce frustration and failure. These children need the reassurance that can be afforded by your attention and by regular and frequent praise. Their need for experiences in successful completion of tasks means that you must be careful and discriminating in what you ask each individual to do. For example, suppose you ask John to pair the members of two sets of objects at the flannel board, and he is unable to manage this task. You then ask Andy to go up to the flannel board to help John. Do not allow Andy to take over and complete the task. Make certain that what Andy does is helping, by doing one pairing only, and that it is John who actually completes the task successfully.

The variability in performance level of children in these classes, which will be discussed in Chapter 4, makes it imperative that you deal with disadvantaged children individually and that you assure each child the experiences of completing his work, with expectations of success, at whatever his current level of performance.

The next two suggestions concern your own language and your encouragement of the children's verbal expression. Given their limitations in auditory discrimination and their inexperience with more complex language structure, they may not be able to understand your language easily. This will be particularly true if you use long, complex explanations or directions. They need short, simply stated directions until such time that they are able to understand more complex verbal expressions, feel more certain in their relationship to you, and you are sure they are able to perform what is being asked of them.

Complementary to the suggestion concerning your language is that of encouraging the children's verbal expression wherever possible. They need the experience of expressing, through words, their ideas and wishes to you as well as to the other children. There are many ways in which this can be accomplished; only a few will be suggested here. By asking children to describe objects (the objects being used for set construction, the toys they are playing with, the pictures they have drawn), you are both encouraging verbal expression and making them aware of attributes or properties (color, texture, size).

Another device which has been found very helpful is the use of word problems. If the teacher starts by telling stories which have problems in them which she expects the children to answer verbally, she can soon get the children to make up such problems for the class to answer.

The last suggestion, keeping the children involved, is certainly applicable to all children. It is especially important for the disadvantaged children. Your knowledge of these children will be of great assistance in this, as will the age of the children. With young children, you can use many sensory-motor experiences for teaching; there can be much more activity involving concrete materials from which abstractions can be made.

In summary, start your teaching at the level at which the children are able to function, use their assets, and maintain a high level of aspiration for yourself and for your pupils.

# Chapter 3

# LANGUAGE AND MATHEMATICAL INSTRUCTION

## I. Introduction

The preceding chapter included a consideration of language characteristics of culturally disadvantaged children and some general implications of these characteristics for teaching these children. In this chapter we shall deal more explicitly with language and mathematical instruction.

In one sense mathematics is a language. It provides a precise means of communicating such ideas as number and space. It has special terms, expressions, and symbolism which, if understood, facilitate its use as a language and which, if not understood, inhibit its use as a language.

A major objective of mathematics instruction in the elementary grades is growth in children's ability to use the language of mathematics effectively. This includes growth both in understanding mathematical concepts and in knowledge of terms, expressions, and symbols associated with these concepts.

Growth in the ability to use the language of mathematics effectively and in the ability to use "general" language effectively are closely related. The development of these should go hand in hand, each reinforcing the other. In this sense, everything you do to improve the general language ability of children can reinforce the development of their mathematical language ability; conversely, everything you do to improve the mathematical language ability of children can reinforce the development of their general language ability.

In this chapter we shall discuss the importance of aural-oral experiences and the learning of concepts, terms, and symbols. We shall consider the preciseness of mathematical language and the use of correct or preferred terminology in teaching young children. Multiple meanings of terms as well as the importance of distinctions between terms will be illustrated by examples from classroom situations.

## II. Aural-Oral Experiences

In keeping with the development of young children's general language abilities, aural-oral experiences precede and receive greater emphasis than do reading-writing experiences in the early stages of the development of mathematical abilities. These listening-speaking experiences need extensive attention.

It is important to note here several kinds of difficulties which children may have in aural-oral learning.

1. Some terms are completely new to children and, if polysyllabic, may not be heard or spoken correctly at the outset. A good illustration of this is the term *equivalent*.

2. Some expressions are relatively lengthy ones, and children may attend to only part(s) of the expression rather than to the whole expression. For instance, with the expression *as many as*, or *as many members as*, children often focus attention on just the "many" or "many members." They do not even sense the word "as" in the expressions.

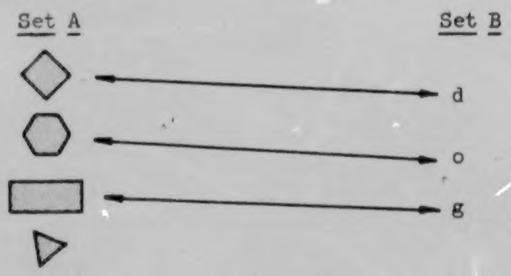
3. Some words sound so much alike that children may fail to sense any distinction between them. Examples of this are the words *six* and *sixth*, or *ray* and *array*. The latter may also be confused with *a ray*.

Difficulties such as these make it imperative that we give most careful attention to speaking and to listening experiences.

A. Setting Language Patterns

We wish to emphasize the "power of example" in developing acceptable mathematical language patterns of young children, --i.e., the power of *your* example.

If terms are to be used correctly by the children, the terms first must be used correctly by the teacher. Consider the following.



In this instance we would *avoid* saying, "Set A is larger than set B." Rather, we would say "Set A has *more members than* set B." We would *avoid* asking, "Which set is larger?" Rather, we would ask, "Which set has more members?"

A child might say, "Set A is larger." Without making an issue of this, we can accept what he has said but respond by giving a preferred language pattern: "Yes, Tommy, set A has more members."

As we work to establish preferred language patterns, it is important that we ourselves enunciate as clearly and as distinctly as possible. This is important at all times, but it is particularly crucial when distinguishing between words such as fifteen and fifty, nineteen and ninety. Failure to enunciate clearly and distinctly can lead to such expressions as "quivun" for "equivalent."

B. Use of Primitive Language

Primitive language often must precede the use of the technical language of mathematics. For instance:

A rectangle is a quadrilateral whose sides determine four right angles (i.e., each pair of adjacent sides determines a right angle). We obviously cannot expect young children to use such terminology from the outset of their work with rectangles!

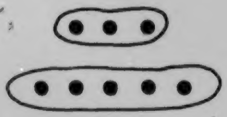
Consequently, with young children we refer simply to the "corners" of a rectangle rather than to the angles associated with a rectangle. Although children certainly could be taught to use the word "angle" from the outset, it would be a meaningless or misunderstood term at that time. Concepts of angle, right angle, and angle associated with a rectangle are much too sophisticated to be used with understanding when young children first work with rectangles. Consequently, at first we have no reasonable alternative but to use primitive terms such as "corner."

Premature use of technical mathematical language, before such terms and expressions can be understood by children, should be avoided.

## III. Concepts, Terms and Symbols

We must distinguish carefully between a child's understanding of a mathematical concept and his ability to use mathematical terms or symbols associated with that concept.

First consider the idea that the order in which two whole numbers are added does not effect their sum, e.g. $3 + 5 = 8$ and $5 + 3 = 8$, or $3 + 5 = 5 + 3$. We refer to this as the commutative property of addition within the set of whole numbers. Children may show in various ways an understanding of this idea. For instance, with the two sets of dots below they may associate either $3 + 5 = 8$ or $5 + 3 = 8$.



However, they may be wholly unaware that they are dealing with a particular instance of the commutative property of addition. An understanding of the concept may exist without knowledge of its technical name. Also, knowledge of the expression, commutative property of addition, in no way indicates or guarantees that children understand the concept.

Now consider the idea of greater than as it applies to whole numbers. Children may know very well that 7 is greater than 4. They may be able to demonstrate this fact by showing that a set of 7 things has more members than a set of 4 things. These same children, however, may not be able to write $7 > 4$, nor understand what $7 > 4$ means when they see it written. An understanding of the concept may exist without knowledge of symbolism associated with that concept.

Our first concern is with the development of mathematical concepts and understandings. Technical terms and symbols are introduced and used only when it becomes advantageous to do so. Frequently this comes much later than the introduction of the concept itself.

A. Preciseness of Language

One characteristic of mathematical language is its preciseness. Consider our use of the expression, is equal to (or, equals).

When we state that "set A is equal to set B" $(A = B)$, we mean simply and precisely that "A" and "B" are names for the same set. When we state that "$3 + 4 = 7$" we mean that "$3 + 4$" and "7" are names for the same number; when we state that "$6 - 1 = 5$" we mean that "$6 - 1$" and "5" are names for the same number. Similarly, when we write "$\overline{AC} = \overline{CA}$" we mean that "AC" and "CA" are names for for the same line segment.

In each of these instances, and throughout our work, the expression is equal to is used to convey precisely the same meaning. We have asserted that one thing--a set, a number, etc.,--has been named in two ways. It is this precise meaning that we convey by the expression, is equal to (or, equals).

### Using the Correct or Preferred Term

The preciseness of mathematical language makes it possible for us to eliminate ambiguity by using correct or preferred terms and expressions.

Consider the statement, "I am bigger than you are." Does this mean that I am taller than you are? Does this mean that I am heavier than you are? The statement clearly is ambiguous in its present form. Any one of these statements would eliminate this ambiguity:

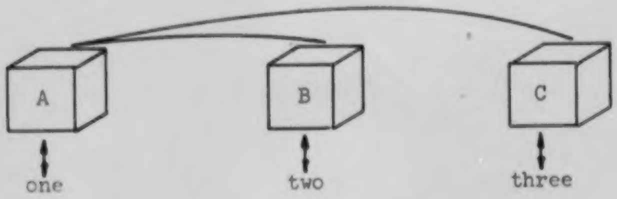
"I am taller than you are."
"I am heavier than you are."
"I am taller and heavier than you are."

In a subsequent section on "Some Important Distinctions," we shall see further illustrations of the fact that using a preferred term or expression eliminates ambiguity. Children need to be helped to choose those words that are unambiguous. One of the best ways of providing this help is by the example we set.

### Multiple Meanings of Terms

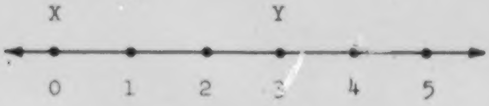
Many words in our language have more than one meaning. We rely upon the context in which such a word is used to suggest the appropriate meaning in a particular situation.

Some mathematical words or expressions also have more than one meaning associated with them. An excellent example of this is seen in connection with counting the members of a set.



For instance, the word "three" may convey either or both of these ideas: it is a label or name for one particular block, block C; it also names the number property of the set whose members are blocks A, B and C. There is a place for each of these interpretations in the development of number concepts.

Consider also a number line representation such as this:



Here we may interpret "3" in either or both of two ways: as a designation or label for point Y, or, as indicating the measure of line segment XY (the segment having X and Y as its endpoints). Again, there is a place for each of these interpretations in the development of mathematical concepts.

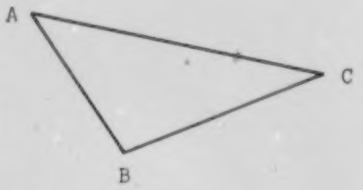
### Some Important Distinctions

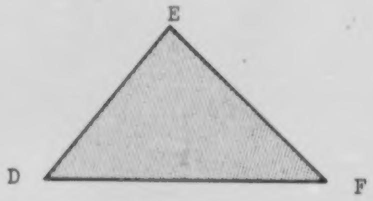
It is not uncommon in mathematical work to distinguish between terms or expressions that are often used as synonyms in "everyday language." Consider these illustrations.

1. In our work we distinguish between the expressions *is equal to* and *is equivalent to*. We previously indicated that if the statement is made, "Set A is *equal* to set B," this implies that "A" and "B" are names for the *same* set. A different meaning is implied, however, by the statement, "Set A is *equivalent* to set B." This latter statement implies that a one-to-one correspondence exists between the members of set A and the members of set B. For each member of A there is a member of B that can be put in correspondence with it,and for each member of B there is a member of A that can be put in correspondence with it. As we use these expressions, *is equal to* and *is equivalent to*, they are *not synonymous*. The fact that two sets which are equal are also equivalent, while two sets that are equivalent may or may not be equal will be discussed in Unit B.

2. In our work we distinguish between a *number* and a *numeral*. For instance, the *number* we refer to as "five" is a property common to a particular class of sets (e.g. the property common to all sets that are equivalent to the set of fingers, including the thumb, of your right hand). The *numeral* "5" is a *name* for that number. The characteristics of *numbers* and *numerals* are quite different.

3. In our work we distinguish between a triangle and a triangular region. For instance, the diagram at the left, below, illustrates a triangle (ABC). It is a union of the line segments $\overline{AB}$, $\overline{BC}$, and $\overline{CA}$.





The diagram at the right, above, illustrates a triangular region. It is the union of triangle DEF and its interior (represented by the shaded portion of the diagram). Clearly, triangle DEF is not the same thing as the region bounded by triangle DEF.

Distinctions such as these necessitate preciseness in the use of mathematical language and are important in clarifying certain concepts and ideas. It is not uncommon that such distinctions involve terms or expressions used synonymously in everyday language, or that they reflect distinctions not made in everyday language.

Different Language Patterns for Different Things

Some ideas which were closely associated with each other may require the use of different language patterns. Consider several ideas concerning sets of physical objects and ideas concerning numbers associated with such sets.

Set A = {a dog, a monkey, a pencil, a bottle}

Set B = {a book, an orange, a trombone}

When speaking of the sets themselves, we say that set A has more members than set B, or that set B has fewer members than set A. When speaking of the numbers associated with each of these sets, however, we say that 4 is greater than 3, or that 3 is less than 4. ($4 > 3$ or $3 > 4$).

If we join sets A and B to show their union, it is appropriate to state that "4 things and 3 things are 7 things." We may associate with this statement the following statement about numbers:

4 *plus* 3 *equals* 7. $(4 + 3 = 7)$.

Here we see a difference between language patterns used with sets of physical objects and language patterns used with numbers.

### B. Using Familiar Meanings

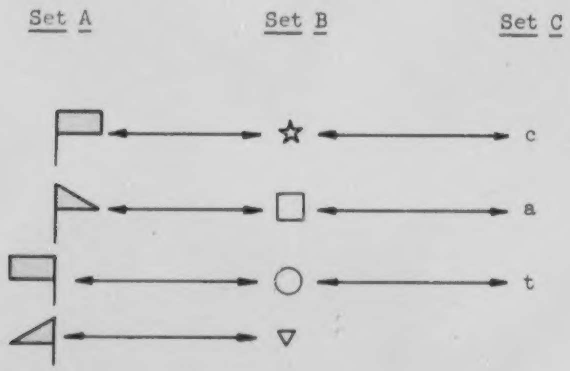
Frequently we can use familiar meanings of words to clarify their interpretation in a mathematical context. One example of this is the word *member*, as it occurs in the expression, "member of a set."

A child often is familiar with the fact that he is a member of his family, or that he is a member of his school class. Such instances of "member" are quite appropriate to use in developing an understanding of a *member of a set*. On other occasions, however, a familiar meaning of a word may not be helpful in developing an understanding of that word as used in a mathematical context. We shall see an illustration of this in the case of the word *match*.

### Special Meanings of Familiar Terms

In their mathematical work children will encounter some familiar words that must be given a special meaning,--a meaning that differs to some degree from one that applies in other contexts. Consider, for instance, the interpretation we attach to the words *pair* and *match*.

Frequently children may be asked to *pair* the members of two sets (in so far as it is possible to do so) to determine whether or not the two sets *match* (i.e. whether or not one set has exactly as many members as the other set). For example:



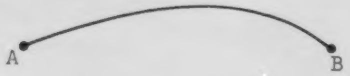
In the illustration on the previous page we can pair each member of set A with a member of set B, we can pair each member of set B with a member of set A. Thus, sets A and B are matching sets (or equivalent sets). There are as many members of one set as there are members of the other. But sets B and C do not match (i.e. they are not equivalent). Although we can pair each member of set C with a member of set B, we cannot pair each member of set B with a member of set C (and use each member of set C only once).

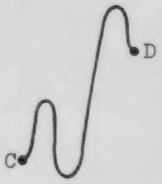
Here we see that a special meaning is given to the words *match* and *matching* as applied to sets. This meaning may differ from the way in which children interpret these words in other contexts. Similarly, *pair* and *pairing*, as applied to members of sets, convey particular meanings to us.

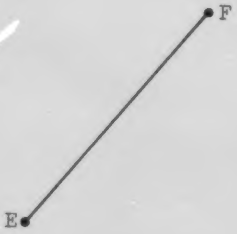
### Language that Contradicts the Vernacular

We have seen that there are occasions when the mathematical interpretation of a familiar term may differ from one or another of its more general meanings. There are other occasions when the mathematical interpretation of a familiar term may even contradict its common meaning.

Consider these three representations.





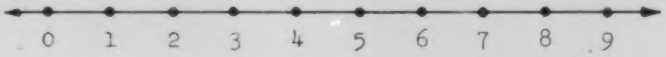


*Each* of these represents a *curve*. The curve at the left has A and B as its endpoints. The curve in the middle has C and D as its endpoints. The representation at the right, with E and F as endpoints, qualifies mathematically as a curve despite the fact that it is "straight." Our use of the term *curve* does *not* imply the idea of "not straight."

Here we have a good illustration of a term whose mathematical interpretation actually contradicts its common meaning under certain conditions.

### C. Non-Mathematical Terms

In teaching mathematical ideas to young children, we must give careful attention to their understanding of words and expressions that are not mathematical but, which are relevant to their learning of mathematics. Consider, for instance, several ideas associated with a number line.



In relation to this number line, the idea of "greater than" is associated with "to the right of," and the idea of "less than" is associated with "to the left of." Specifically, 8 is greater than 5, and the point labeled 8 is to the right of the point labeled 5. Also, 3 is less than 7, and the point labeled 3 is to the left of the point labeled 7. Thus, it is crucial that children be able to distinguish between the non-mathematical terms right and left in order to interpret the mathematical relations of greater than and less than in using the "number line." Young children often have difficulty with the right-left distinction, but it must be mastered if certain mathematical ideas are to be understood. Therefore, we must give attention to those non-mathematical terms that are essential to the development of mathematical concepts.

### D. Unnecessary Terms

Contemporary approaches to the teaching and learning of mathematics make unnecessary certain terms that in the past were a familiar part of elementary school arithmetic vocabulary. Good illustrations of this are the terms minuend, subtrahend, multiplier, and multiplicand.

Consider these number sentences: $3 + 5 = 8$ and $8 - 5 = 3$. In each instance 3 and 5 are addends, and 8 is their sum. It is not necessary to use one set of terms with the addition example and another set of terms with the subtraction example.

Also consider the number sentence, $3 \times 5 = 15$. In this instance 3 and 5 are factors whose product is 15. If we wish to distinguish between the two factors, we may refer to 3 as the "first factor" and to 5 as the "second factor." There is no need for the terms multiplier and multiplicand.

Contemporary approaches to the teaching and learning of mathematics have introduced some new terms; on the other hand, contemporary approaches to the teaching and learning of mathematics have made unnecessary certain terms that were used in the past, but frequently not understood.

## IV. Concluding Statement

Mathematics is a language. The teaching and learning of mathematics is therefore, the teaching and learning of a language.

The language of mathematics has its unique concepts, terminology, and symbolism. In this chapter we have attempted to highlight some of the elements that appear to be particularly crucial for culturally disadvantaged children as they learn to use this language effectively.

If we were to single out any one thing that is most important in this connection, it would be the power of the teacher's own example. Children's learning of mathematics as a language will be advanced in direct relation to the strength of the language model that you, as their teacher, set for them. It is our hope that the inservice experiences in this course will increase the strength of that model.

## Chapter 4

## NUMBER CONCEPTS OF DISADVANTAGED CHILDREN

### Description of the Study

Observation classes at the kindergarten and first grade levels were established for the 1964-65 school year in Boston, Chicago, Detroit, Miami, Oakland, and Washington, D. C. in areas described as economically and culturally disadvantaged.

This study was based on the idea that there is a differential in experiences prior to school entrance between middle-class and lower-class children. Studies from a number of sources suggest that children from more advantaged homes tend to have had experiences of greater variety in an organized family setting. By the time these children reach school-age, they appear to be better able to work in a group situation, to utilize verbal skills, and to deal with abstract concepts.

One aim of the study was to gather information on the stage of development of certain concepts relevant to the learning of mathematics in these children at the beginning of the school year and to study their growth during the year. Another aim was to discover what mathematical concepts caused the children difficulty so that a more effective program could be developed for them and to provide information to help develop materials for teachers emphasizing techniques for providing disadvantaged children with experiences necessary to make the program more effective.

The procedures adopted for gathering the information were varied and included individual and small group testing of the pupils, weekly reporting by the teachers, observations of the classrooms, and four conferences of the participating teachers.

### Testing

Individual tests were administered to the pupils in October, January, and May. A pencil and paper group test was given in June to each class. The class was split into several small groups to make it easier to administer this particular test.

The table below shows what assessments were made and at which testing session for both the kindergarten and first grade pupils.

Description of Tests

| Scores Obtained | Initial | | Mid-Year | | Final | |
|---|---|---|---|---|---|---|
| | K | 1st | K | 1st | K | 1st |
| Object Recognition | X | X | | | | |
| Photograph Recognition | X | X | | | | |
| Drawing Recognition | X | X | | | | |
| Vocabulary | | | X | X | | |
| Visual Memory-Objects | X | X | | | X | X |
| Visual Memory-Pictures | | | | | X | X |
| Color Inventory-Matching | X | X | | | X | X |
| Color Inventory-Naming | X | X | | | X | X |
| Color Inventory-Identifying | X | X | | | X | X |
| Geometric Shapes-Matching | | | X | X | | |
| Geometric Shapes-Naming | | | X | X | X | X |
| Geometric Shapes-Identifying | | | X | X | X | X |
| Pairing | | | | X | | |
| Equivalent Sets | | | X | X | | |
| Counting Buttons | X | X | X | X | X | |
| Counting Sets | | | X | X | X | X |
| Recognition of Number Symbols | X | X | | | X | X |
| Marking Number Symbols | X | X | | | X | X |
| Place Value-Naming | | | | | | X |
| Place Value-Forming | | | | | | X |
| Ordinal Number | X | X | | | X | X |
| Rote Cardinal Counting | X | X | | | X | X |
| Rote Cardinal Counting by Tens | | | | | | X |
| Ordering | | | X | X | X | X |
| Classifying | | | X | X | X | X |
| Response to Verbal Directions | X | X | X | X | X | X |
| Attention to Tasks | X | X | X | X | X | X |

In the space available it is not possible to include all the results from each session. A selection has been made of those results that highlight certain points in the learning of mathematical concepts of these disadvantaged children. The specific areas included are object recognition, color, geometric shapes, rational counting and the recognition and writing of number symbols. These are typical of the level and range of abilities found in all the classes tested and point out clearly some of the problems that the teachers of disadvantaged children encounter.

## Object Recognition

In this section, the pupil's ability to recognize objects and to recognize pictorial representations of such objects that are used in the curriculum materials, was measured. The child was shown an object, a drawing of an object, and a photograph of an object and asked, "What is this?" For object recognition (23 items), the mean score was approximately twenty; for photograph recognition (10 items), nine; and for drawings (7 items), it was approximately six. Little difference was found between the classes in the disadvantaged areas and classes in middle-class areas.

Objects causing difficulty were different fruit, coins, and string. From 10-20% of the pupils were unable to name orange; banana 10%; penny 10%; nickel 20-30%; dime 20-25%; and string 20-40%. Although many could not name string, they could indicate what function it served.

The results indicate that most pupils are able to name and identify the objects suggested in the text materials for use in the classroom.

Do not infer from these results, however, that the verbal skills and experiences of the two groups are the same. It has already been pointed out that children from disadvantaged groups will lack many of the experiences which facilitate school learning.

## Color and Geometry

Children from all backgrounds are able to match the basic colors, but their ability to name these colors and to select a color when given its name are very variable at the beginning of kindergarten. Children from middle-class areas are fairly proficient on these two tasks when entering first grade but children from disadvantaged areas are not. Typical results for classes in the same city were:

|  | Number of Items | KINDERGARTEN<br>E | KINDERGARTEN<br>C | FIRST GRADE<br>E | FIRST GRADE<br>C |
|---|---|---|---|---|---|
| Matching | 6 | 5.3 | 5.5 | 5.5 | 6.0 |
| Naming | 7 | 5.1 | 5.8 | 5.8 | 6.8 |
| Identifying | 6 | 4.6 | 4.7 | 5.0 | 5.9 |

[E denotes an experimental class (disadvantaged area) and C denotes a control class (middle class).]

The results from this and other inventories point out the need for teachers to be aware of the three-fold nature of many of the tasks that children must learn at these grade levels. The child has to be able to match two objects, e.g. the name "three" to the numeral "3" and, when given a number name, to be able to select the correct number of objects. The assessments showed generally that matching was the easiest, then identifying, and naming the most difficult. However, by continually providing the children with experiences in the three phases, considerable improvement can be seen. The results in geometry for the same first grade classes from the mid-year to final inventories show this.

GEOMETRIC SHAPES - PERCENTAGE CORRECT RESPONSES

First Grade

|  | Experimental<br>Mid-Year | Experimental<br>Final | Control<br>Mid-Year | Control<br>Final |
|---|---|---|---|---|
| Matching |  |  |  |  |
| Circle | 96 | -- | 100 | -- |
| Square | 96 | -- | 96 | -- |
| Triangle | 96 | -- | 100 | -- |
| Rectangle | 96 | -- | 100 | -- |
| Mean | 3.8 |  | 4.0 |  |
| Naming |  |  |  |  |
| Square | 39 | 81 | 93 | 93 |
| Triangle | 57 | 96 | 93 | 79 |
| Rectangle | 0 | 50 | 85 | 64 |
| Circle | 50 | 73 | 81 | 89 |
| Mean | 1.4 | 3.0 | 3.5 | 3.3 |
| Identifying |  |  |  |  |
| Triangle | 89 | 100 | 93 | 89 |
| Rectangle | 14 | 77 | 85 | 89 |
| Circle | 96 | 92 | 96 | 100 |
| Square | 64 | 85 | 93 | 96 |
| Mean | 2.6 | 3.5 | 3.7 | 3.8 |

This table shows the gains that pupils in the experimental first grade class were able to make in naming and identifying the geometric shapes. These gains are more clearly seen in the table below which shows the frequency distribution of correct responses for the same classes in naming and identifying the geometric shapes. It should be noted that although the pupils improved, there remained more variability within the experimental class.

NAMING AND IDENTIFYING GEOMETRIC SHAPES

| Number of tasks | Naming | | | | Identifying | | | |
|---|---|---|---|---|---|---|---|---|
| Successfully completed | Mid-Year | | Final | | Mid-Year | | Final | |
| | E | C | E | C | E | C | E | C |
| 0 | 6 | 0 | 0 | 0 | 1 | 2 | 0 | 0 |
| 1 | 6 | 0 | 3 | 0 | 1 | 0 | 0 | 0 |
| 2 | 13 | 2 | 3 | 8 | 8 | 2 | 4 | 2 |
| 3 | 3 | 9 | 11 | 5 | 15 | 1 | 4 | 2 |
| 4 | 0 | 16 | 9 | 15 | 3 | 23 | 18 | 24 |

## Rational Counting

Several different tasks were used to measure the pupil's ability to count given sets of objects and to select given numbers of objects. Other number tasks such as rote cardinal counting were also included in the assessments. There is little evidence to suggest however that a child who can count by rote will necessarily count well rationally.

One task pupils were asked to do was to count out 3 buttons, then 5 buttons, 4 buttons, 6 buttons, 8 buttons, 7 buttons, and 9 buttons. Initially, the average kindergarten pupil could complete successfully only two or three of these tasks. At the end of the school year, this pupil could manage about six tasks correctly. Similar improvement was found with first grade pupils from the initial inventory (four to five of these tasks correct) to mid-year when the average was almost seven correct. The greatest difference found between the experimental and control classes was the greater variability among students in the experimental classes.

COUNTING BUTTONS

| Number of tasks Successfully completed | Kindergarten Initial E | Kindergarten Initial C | Kindergarten Final E | Kindergarten Final C | First Grade Initial E | First Grade Initial C | First Grade Mid-Year E | First Grade Mid-Year C |
|---|---|---|---|---|---|---|---|---|
| 0 | 2 | 3 | 0 | 0 | 2 | 0 | 2 | 0 |
| 1 | 12 | 3 | 0 | 3 | 0 | 0 | 1 | 0 |
| 2 | 4 | 1 | 0 | 0 | 2 | 0 | 0 | 1 |
| 3 | 5 | 2 | 2 | 0 | 1 | 0 | 0 | 0 |
| 4 | 2 | 6 | 1 | 0 | 2 | 2 | 0 | 0 |
| 5 | 3 | 2 | 1 | 0 | 0 | 5 | 1 | 1 |
| 6 | 1 | 1 | 2 | 3 | 6 | 0 | 4 | 1 |
| 7 | 1 | 6 | 20 | 17 | 16 | 21 | 20 | 24 |

The variability in level of performance between pupils from disadvantaged areas is not confined to number tasks but is also evident in other tasks that they were set. The table below shows the frequency distribution of correct responses on color naming for two classes of first grade children at the end of the year.

NAMING COLORS

First Grade

| Number Correct | Experimental | Control |
|---|---|---|
| 0 | 1 | 0 |
| 1 | 0 | 0 |
| 2 | 0 | 0 |
| 3 | 0 | 0 |
| 4 | 3 | 1 |
| 5 | 4 | 1 |
| 6 | 10 | 0 |
| 7 | 11 | 26 |

### Recognition and Writing Number Symbols

First grade teachers often assume that most of their pupils can recognize, name, and also write many of the numerals when they start first grade. However, children in disadvantaged areas will generally not be able to do any of these tasks well, as the tables show.

## RECOGNITION AND WRITING NUMBER SYMBOLS - PERCENTAGE CORRECT

| Numeral | Recognition | | Write | |
|---|---|---|---|---|
| | E | C | E | C |
| 0 | 79 | 100 | --- | --- |
| 3 | 83 | 100 | 48 | 79 |
| 4 | 76 | 100 | 24 | 100 |
| 5 | 76 | 100 | 21 | 100 |
| 6 | 65 | 100 | 31 | 89 |
| 7 | 69 | 100 | 24 | 68 |
| 8 | 69 | 100 | 45 | 89 |
| 9 | 59 | 100 | 14 | 79 |

## FREQUENCY DISTRIBUTION

| Number Correct | Recognition | | Write | |
|---|---|---|---|---|
| | E | C | E | C |
| 0 | 2 | 0 | 10 | -- |
| 1 | 3 | 0 | 5 | -- |
| 2 | 1 | 0 | 2 | -- |
| 3 | 1 | 0 | 5 | 2 |
| 4 | 0 | 0 | 2 | 2 |
| 5 | 0 | 0 | 2 | 4 |
| 6 | 3 | 0 | 3 | 5 |
| 7 | 3 | 0 | 0 | 15 |
| 8 | 16 | 28 | -- | -- |

Although the recognition and writing of numerals are NOT fundamental to the understanding of mathematical concepts, these are skills essential for communicating mathematical ideas and concepts, and the child who lacks them will be handicapped for future learning.

All the pupils will need additional practice in writing numerals, but this practice should not be given until they have an understanding of the elementary concepts of number. It cannot be assumed that if children can recognize the numerals they can write them. They will need a great deal of careful practice which can be given independently of the mathematics lessons.

In the early stages an adhesive number line attached to pupils' desks is a very useful aid as they can use it whenever they need it. Even with number lines on their desks there may be some pupils who will not be able to form the numerals correctly, e.g. "7" will be made as " " and "9" as " " and the pupils will need additional assistance.

### General Discussion

One fact that clearly emerges from the assessments made this year is that it was difficult to predict from the initial test what development and progress a pupil will make during the year. Some pupils whose scores on the initial assessment were low, progressed very rapidly; others did not. All pupils did make progress but at variable rates. This is one of the major problems that confronts the teacher of disadvantaged children. The range of abilities in these classes is much wider than found in classes of middle-class children, as has already been noted.

A technique which the teachers found useful this year was to observe carefully three selected pupils each day. These systematic observations gave the teacher valuable information about the strengths, weaknesses and difficulties of each pupil and enabled her to chart changes in his level of performance.

In attempting to meet this variability, some teachers grouped pupils for instruction as is often done in reading, while others kept the class as a unit but provided work sheets which took into account the different performance levels of the pupils. If grouping is used, it is essential to move an individual from group to group as he changes in his level of performance. The success of any method depends upon the pupils being able to use blocks or other manipulative material to do their work sheets while the teacher works with a small group or an individual.

As yet there are no adequate materials nor work sheets that will keep all of the pupils fully and profitably engaged at all times. This is a problem and challenge that each teacher must face. However, a great deal can be done and a great deal more can be learned if each teacher tries a variety of approaches and reports her successes and failures.

# UNIT B

## CHAPTER 1

## SETS

### Introduction

There are a few ideas that occur over and over again in mathematics; one of these is the concept of sets. This concept occurs, for example in dealing with sets of points, sets of numbers, sets of objects. The most general of these of course, are sets of objects, and it is from these sets that we ultimately elicit the concept of numbers. Thus, sets help form a primitive basis for the number concept and serve as pre-number ideas.

### What is a Set

In speaking of collections of objects, special words may be used with reference to special collections such as herd for cattle, flock for geese, bevy for quail, pride for lions, set for dishes, and so on. Each of these may be equally described as a set; a set is just a collection of things. The furniture in the room, the monkeys in the zoo, the doors in a room, the children in the class are each a set of things. Each object in a set is called a member or an element of the set. If the objects on your desk are a pencil, a book, a calendar, and a blotter, then each of these is a member of the set of things on your desk; each child in your class is a member of the set of children in your class.

As we can see, a set may consist of a variety of objects. A prime characteristic of a set is that of being well-defined. That is, there should be a way of determining whether a particular object is or is not a member of a particular set. For the most part, the determination is quite clear. For example, in the case of the set of children in this classroom, precluding unusual circumstances, Rosa is either a child in the classroom or she isn't; in the case of the set of objects on your desk, the criterion for determining whether a particular object is a member of this set is, "Is this object on your desk?" In either event, there is a property that is shared by members of the set that is not shared by objects that are not members of the set. The common property of being a child in this classroom thus specifies a particular set. On the other hand, a set may consist of quite unrelated objects; for instance, we may wish to discuss a set consisting of George Washington and the North Pole. Then, the common property that specifies the set may be no more than simply being members of the set. Eddy is not a member of the set consisting of George Washington and the North Pole; baseball is not a member of the set consisting of George Washington and the North Pole. In this manner, the members of the set consisting of George Washington and the North Pole are specified; in turn, the set is specified or well-defined. In short, if a set is to be well-defined, then for any object, we should be able to answer the question, "Is this object a member of the set?"

An example in which the set is not well-defined is a set of children; it is not clear by this incomplete description whether or not a particular child is a member of this set. Another example in which the set is not well-defined is the set of all good children in this classroom. Here, it is not a question of the description being incomplete as it is that subjective judgment enters into the decision about a particular child's membership in this set.

Problems *

1. What are the members of the set of
    a. the Great Lakes of the United States
    b. the musical notes in the major triad
    c. the objects in Elsie's purse
2. Which of the following sets are not well-defined if they are
    a. all thoroughbred French poodles
    b. all pretty women
    c. all letters of the English alphabet
    d. all physics books

## Describing Sets

There are various ways in which a set may be specified. As in the case of George Washington and the North Pole, we may specify the set by listing all the members. A class roster is thus a means of specifying a particular set; a reading list is a means of specifying another set. If the reading list consists of the book titles, The Story of Ping, A Day in Maine, and Make Way for Ducklings, we can enclose these titles within braces to denote the set so specified. Thus,

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* Solutions for problems in this chapter are on page 56.

{*The Story of Ping*, *A Day in Maine*, *Make Way for Ducklings*} is a notation for "the set whose members are *The Story of Ping*, *A Day in Maine*, and *Make Way for Ducklings*." The braces are an abbreviation for the words "the set whose members are." Note that the items in the listing are separated by commas.

There are occasions when it is inconvenient or impractical to specify the set by listing all its members. For example, the set of all states of the United States requires a listing of 50 states; the set of all inhabitants in the United States may require a listing of more than 200 million names. If there is an explicit common property that may be used to characterize the members of the set, then such a description may be adequate. Thus,

{the states of the United States}

specifies the set being considered. For convenience, we may use a letter symbol to label a particular set, and once so identified, refer to this set by its label. Thus, if we agree to label the set of states of the United States by the letter A, then we can write

A = {states in the United States}.

Thereafter, the set of states in the United States may be referred to simply as A. Conventionally, upper case letters are used for this purpose.

We have mentioned that a class roster is a means of specifying a particular set. Note that a child's name is not listed more than once in specifying the set. Once he is listed, he is designated as a member of the set. By the same token, {d, e, r} is the set of all letters in the word "deer" as well as in the word "red".

### Problems

3. Using a common property, describe the set specified by
   a. {Alaska, Hawaii, Washington, Oregon, California}
   b. {Maine, Vermont, New Hampshire, Massachusetts, Rhode Island, Connecticut}
   c. {red, yellow, blue}

### Equal Sets

When we write

A = {states in the United States}

we mean that A and {states in the United States} are symbols or names for the same thing. Whenever we use the equal sign "=" as in

$$5 + 2 = 7$$

we shall mean that the two sets of symbols are names for the same thing; in this case "5 + 2" and "7" are both names for the same number.

Note that {the first 5 letters of the English alphabet} is identical with {a, b, c, d, e}. To indicate that we have one and the same set, we say that these are equal sets and we write

{the first 5 letters of the English alphabet} = {a, b, c, d, e}.

In other words, if A is a set and B is a set, then

$$A = B$$

if each has the same members as the other.

As another illustration, since the set consisting of the members Rosa, Eddy, and Leon is identical with the set consisting of the members Eddy, Rosa, and Leon, we can write

{Rosa, Eddy, Leon} = {Eddy, Rosa, Leon}.

In view of this, note that the order in the listing of the elements is immaterial in specifying the set; the same set is specified by two different listings of the same members.

## Problems

4. Are any of the following four sets equal?
   A = {1, 3, 5}
   B = {numerals representing the first three positive odd numbers}
   C = {135}
   D = {9}
   E = {the digits in the numeral 1351}

## Subsets

A set is a collection of elements. The selection of certain elements from a given set will form a set. For example, from

$$A = \{a, b, c, d, e\}$$

we may form a set consisting of the elements, a, c, d:

$$B = \{a, c, d\}.$$

With reference to A, we say that B is a subset of A. A set, B, is said to be a subset of a set, A, if each element of B is also an element of A. Thus,

{Rosa, Eddy} is a subset of {Rosa, Eddy, Leon}

because each member of {Rosa, Eddy} is a member of {Rosa, Eddy, Leon}. However,

{Rosa, Anthony} is not a subset of {Rosa, Eddy, Leon}.

Observe that if

$$A = \{a, b, c, d, e\} \text{ and } B = \{a, b, c, d, e\}$$

then it is true that every element of B is an element of A; so B is a subset of A. Since $A = B$, this example illustrates that one of the subsets that may be formed from a given set is simply the given set. This should not be too surprising considering that one of the subsets that can be built from a given set of elements is precisely the set consisting of all the given elements. Indeed, this may be so taken for granted that the need to make such a statement is not at all apparent. However, this fact will have some undertones for us, as for example, when we examine certain special cases for subtraction.

There are times when we want to differentiate the subset that is identical with the given set from one that is different from the given set. When this is indicated, we call a subset that is different from the given set a proper subset. Accordingly, the subset that is the same as the given set is called an improper subset. Thus,

{a, b, c} is a proper subset of a {a , b, c, d, e},
{Rosa, Eddy} is a proper subset of {Rosa, Leon, Eddy},

and {a, b, c, d, e} is an improper subset of {a, b, c, d, e}. In other words, B is a proper subset of A if B is a subset of A and there is at least one element of A that is not an element of B.

We have noted that if

$$A = \{a, b, c, d, e\} \text{ and } B = \{a, b, c, d, e\}$$

then B is a subset of A; here, it is equally true that A is a subset of B. We can also see that

if A is a subset of B and if B
is a subset of A, then $A = B$.

Problems

5. Which expression states that the letter y is an element of the set of letters in the word "Friday"?
   a. y is an element of {Friday}
   b. {y} is an element of {Friday}
   c. y is an element of {F, r, i, d, a, y}
   d. {y} is an element of {F, r, i, d, a, y}

6. {Tom, Harry} is a subset of {Tom, Dick, Harry}. Name all eight different subsets of {Tom, Dick, Harry}.

## A Set with One Member and the Empty Set

The set of all vowels in the word "cat" consists of just the member, a. That is to say,

{the vowels in the word "cat"} = {a}.

This is an example of a set with one member. It may conflict with our intuitive sense to think of a set with a single member since, in ordinary language, the word "set" connotes more than one object in the collection. An even more bizarre set that we shall describe is the set that has no members. Both of these mathematical concepts--of a set with one member and of the set with no members--are handy ones; moreover, a vital question of logic requires the existence of such entities.

Logically, unless the concept of a one-member set is considered, it would make no sense to come up with "a" as the set of all vowels in the word "cat"; the letter "a" does not answer the question: "What is the set of all vowels in the word 'cat'?" Likewise, the same question of logic may enter into the consideration of the set with no members. A plea may be made that the question itself needs to be reworded. Instead of asking, "What is the set of vowels in the word 'cat'?", it may be more appropriate to ask, "What is the vowel in the word 'cat'?" This may sound sensible, but it does require a prior knowledge of the answer. Quite often, we do not know how many solutions we may have to a problem. With the understanding that there may be one, more than one, or no members in a set, there would be no need to rephrase the question each time a special situation is encountered. For example, the question, "What is the set of boys enrolled in this school?" might be equally applicable to the Yale, Columbia and Vassar populations--or to one in which just one boy happens to be enrolled.

In thinking about a set with one member, there is a strong inclination to think of the set and the member that constitutes the set as one and the same thing and it is important to distinguish between the two. A case in point might be given, for example, in the cataloguing of books in the school library. Under the category of classics might be just the one book, *Treasure Island*. By itself, the book is not the same as the set of classics. If another book is added to the collection, the set of classics has changed; *Treasure Island* has not changed.

*The empty set* is the set with no members. Thus, the set of all boys enrolled in Vassar is an example of the empty set. The set of all months having nine Sundays is another example of the empty set. There may be many ways of illustrating the empty set but any example of the empty set has the same members as any other example of it because none of them has any members. This is why we say *the* empty set; there is only one such set. A notation for the empty set is { }. The empty space between the braces indicates that there are no members in the set. Another notation that is used for the empty set is the symbol $\emptyset$. With the first way of denoting the empty set the question may arise as to whether we had forgotten to list the elements within the braces. With the second way, there is not this problem.

Recall that B is said to be a subset of A if each member of B is also a member of A. Another way to say this is

> B is a subset of A if there is no member of B which is not also a member of A.

Both statements say exactly the same thing. As a consequence of the second statement, the empty set is a subset of

A = {Rosa, Eddy, Leon, Anthony}.

There is no member of { } that is not also a member of A. The empty set has no members. Thus, the empty set is a subset of every set.

*Problems*

7. Which of the following are equal sets?

A = {a, e, i, o, u};

B = { };

C = {Monday

D = {women who are 20 feet tall}

E = {the vowels in the English alphabet}

F = {the days of the week}

8. Which of the sets in the above list is a proper subset of another in the list?

Applications to Teaching

While both symbols, $\emptyset$ and { }, have been used here to denote the empty set, it is best to avoid introducing too many symbols simultaneously. Since the braces have been used for sets consisting of many members as well as for sets with one member we have kept to the use of the braces, { }, for the empty set for students in the primary grades. This notation does have the advantage of suggesting no members in the set.

As indicated above, the set with one member and the empty set may not seem to be easy concepts to present. Many teachers have, however, reported that children have been able to grasp these concepts quite easily. Since these sets will ultimately be associated with the numbers 1 and 0, they need to be included in our experiences with sets. There should be emphasis on the use of the article "the" in referring to the empty set. As in many other instances, for this level, the emphasis is largely by precept and example on the part of the teacher; there needs to be constant awareness of the proper use of language.

Both the proper use of language and the deliberate stress on certain critical terms are particularly important in view of the listening habits of some children. Some may not be able to grasp all that is said in long expressions. Some will attend to only part of what is said. Thus, aside from the cavalier reference to "the empty set" as "an empty set", there may be confusion between the words "set" and "subset". Unless conscious effort is made in enunciation, these terms may sound alike to the youngsters. In addition to marked effort in the proper use of language, constant and natural use of new terms throughout the day as occasions may arise has been found to be helpful. For example, there may be many instances of subsets that can be pointed out; during play period,

"a subset of the class that is on team A",

during reading,

"a subset of the ducklings in the pond",

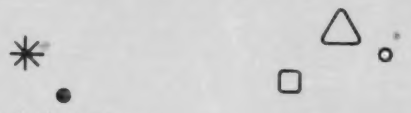
during music,

"a subset of the class that is playing the piano",

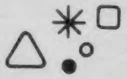
and so on.

Team membership offers excellent reinforcement of the fact that rearranging the members does not change the set. This is not an easy concept to teach and a variety of experiences may need to be provided leading to this notion. Some children are quite convinced that each time there is a new arrangement of the same members, a new set is formed. By way of illustration we might mention that the same members make up the set (team) regardless of who is on first base, who is on second and so on. If there is any change in membership, a different team is actually formed. Another illustration may be given in changing seat assignment in the classroom; the same set (class) of students is in each arrangement. Books may be arranged differently on a shelf. If there is no change in membership, each arrangement gives us the same set of books. The students will be faced with this concept again and again. For example, when we begin to compare sets of objects or when we partition a set into subsets to arrive at the concept of division, there will be opportunity to reinforce this notion.

We want to communicate the concept that a set is defined by the members; it does not matter how widely spaced these members may be. For example,



is the same set of objects as



Again, the illustration of team membership may be helpful. Initially, we may "choose sides" by grouping the members of each team together. Once the team membership is determined, the same members constitute the same team regardless of location of the individuals. The set of classics belonging to the school library, for example, may be defined by the cataloguing. Some of these books may be clustered in groups for a display; some may be on various shelves; some may be out on loan; spatial arrangement is immaterial to defining the set.

Our goal, through this discussion, has been to emphasize that set membership is independent of spatial arrangement. However, we recognize the intuitive aspects in visual perception. In a visual display, the spatial arrangement of a set of objects may suggest a natural grouping.

Thus the arrangement

```
x x x x x   x x x x x   x x x x x
x x x x x   x x x x x   x x x x x
```

might suggest 3 groups of ten objects. Later on, when we examine the basis underlying our numeration system, we do capitalize on this tendency to group on the basis of spatial arrangement. For example, to arrive at a particular decimal numeral, a set of objects may be spatially grouped into subsets of 10's and 1's, and so on.

As was mentioned in the introduction, we shall ultimately elicit the concept of numbers from sets of objects. In particular, we shall associate a set with one member with the number 1. While it is true that a set with one member cannot be considered (from the standpoint of logic) before a person has a concept of the number 1, nonetheless, it will be found that almost universally the children will already have "one" in their vocabulary. The word "single" may cause difficulty for some children. From a teaching standpoint, we may rely on using the words "one" and "a single" interchangeably to communicate some needed concepts. Again, classes react differently to the situation. Some teachers report success <u>because</u> the word "single" has been foreign to the students' vocabulary.

Along with emphasizing the natural use of language, we would like to emphasize the natural presentation of topics. By this, we mean a de-emphasis on fiat; these are the things you must learn and this is the way you learn them! By natural presentation, we also mean minimizing forced-feeding. At times, it may appear that teaching certain concepts reaches an impasse. Subsequently, in conjunction with presentation of some different topic, some students's remarks may reveal that what had appeared to be an impasse before is no longer one. It is likely that some incidental learning has occurred. It is also likely that there is reinforcement with other disciplines that, together with the presentation in mathematics, help to bring the concepts into focus. There is little need to insist on complete mastery immediately. Oftentimes, it is best to proceed with other developments when an impasse is apparent and return to the topic sometime in the future.

## Exercises - Chapter 1

1. List the elements of each of the following sets whose descriptions are:
    - a. {the days of the week whose names begin with the letter W}
    - b. {positions on a baseball team}
    - c. {months of the year whose names have less than six letters}
    - d. {whole numbers between 7 and 8}
    - e. {the age on the nearest birthday for the students in your classroom}
    - f. {the capitals of Japan and England}
    - g. {the colors of the rainbow}

2. Write a description of the set:
    - a. {gh, ph, f}
    - b. {snips, snails, puppy-dog tails}

3. Describe the common property of the elements of { ☆, pencil, red}.

4. Which of the following pairs of sets are equal?
    - a. {17} and {71}
    - b. {letters in the word bundle} and {n, d, b, l, u, e}
    - c. { *, q, ϕ} and {q, * , ϕ }
    - d. {zero} and {peacocks native to the North Pole}
    - e. {1, 2, 3, 4, } and {a, b, c, d }
    - f. {are} and {era}
    - g. {M, i, s, p} and {the letters in the word "Mississippi"}

5. For each of the following, decide whether the statement is true or false and why.
    - a. 3 is a subset of {1, 2, 3}.
    - b. {ego} is a proper subset of {ego, je, I}.
    - c. It is possible for a set to be equal to one of its subsets.
    - d. {all birds in the world} is a subset of all {hens in the world}.

6. Given the set A = {rose, bee, tulip, beetle, dandelion}, write the subset of A described by
    - a. {plants}
    - b. {insects}
    - c. {singers}

## Solutions for Problems

1. a. Erie, Huron, Michigan, Ontario, and Superior.

   b. do, mi, sol or root, major third, perfect fifth or tonic, mediant, dominant.

   These lists are not different sets. They are simply different ways of naming the same elements. There may be others, but the names must be general; e.g., C, E , G would not be a correct list of elements.

   c. This set is not well-defined. The set of objects depends on which Elsie. Even if Elsie is uniquely identified, it will be agreed that the set of objects changes in time. In order to specify the set (make it well-defined), it is necessary to consider a particular Elsie at a particular moment.

2. a and c *are* well-defined.

   b. This set is *not* well-defined. Subjective judgment is involved in deciding whether a particular woman belongs to the set or not. Specification is further complicated by the fuzzy distinction made between girls and women in our society. Is a pretty eighteen year old to be included?

   d. The set of all physics books is not well defined. Would a book on vector analysis, one on physical chemistry or one on the history of physics be included?

3. a. {all states of the United States which border on the Pacific Ocean}

   b. {the New England states}

   c. {the primary colors}

4. Sets A, B and E are equal. No others are equal. C is a single member set whose element is the numeral for one hundred thirty-five. The fact that no commas separate the digits, makes it different from 1, 3, 5. D contains the numeral for nine. Even though the sum of 1, 3, and 5 is 9, 9 itself is not another name for 1, 3, 5. D would be equal to {1 + 3 + 5} and 1 + 3 + 5 would be the single element of the set. The reason E is the same as A and B is that the digits of the numeral 1351 are 1, 3, 5 and 1. Recall, however, that an element is *not* repeated in a set, so the 1 should only be mentioned once.

5. c. only. a. is incorrect because {Friday} is a single member set whose element is the name of the fifth day of the week, which is not y. b and d. are incorrect because {y} is a set. Neither {Friday} nor {F, r, i, d, a, y} have any members which are themselves sets.

6. { }, {Tom}, {Dick}, {Harry}, {Tom, Dick}, {Tom, Harry}, {Dick, Harry}, {Tom, Dick, Harry}

7. A = E; B = D

8. C is a proper subset of F.

# CHAPTER 2

# COMPARING SETS

## Basis of Comparison

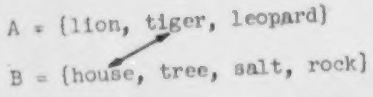
One of the ways that we have used to specify a set is to describe it by the property that the elements have in common. This method of classifying things can be extended to help distinguish one kind of set from another. Associated with this is the question: "What characteristic does one set have in common with another set?" Essentially, this is a classification problem that is one step removed from identifying the common property of elements within the set. For example, while the sets

> A = {lion, tiger, leopard} and C = {elephant, deer, cow, horse} are not equal, both of these are sets of animals, and may be distinguished from B = {house, tree, salt, rock}.

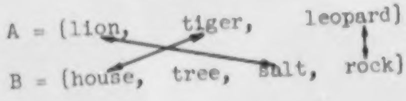
A further distinction might be that A is a set of carnivorous animals and C is a set of herbivorous animals. The point is that sets may be compared one with another.

## One-to-one Correspondence

One way of comparing two sets is by an element-by-element pairing. That is, an object of one set is paired with an object of the other set. To indicate a pairing we shall draw a double-headed arrow between the two members. Thus,



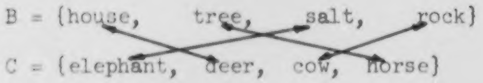
shows that the element "tiger" is paired with the element "house". The diagram



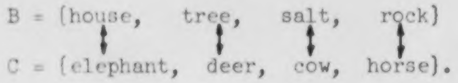
indicates that:

| | | |
|---|---|---|
| lion | is paired with | salt; |
| tiger | is paired with | house; |
| leopard | is paired with | rock. |

Another illustration of a pairing may be given by


B = {house, tree, salt, rock}
C = {elephant, deer, cow, horse}

For our purpose, the concern is not so much that "lion" is paired with "salt" as that one member of A is paired with one member of B. Notice that in pairing the elements of B with those of C, each element of B is paired with an element of C and each element of C is paired with an element of B. When this happens, then we say that the sets match; also, we say that we have a one-to-one correspondence between the elements of the two sets. It can be seen that whether we can get a one-to-one correspondence between the elements of two sets does not depend on which element of B is paired with which element of C. For example, the pairings may be established by


B = {house, tree, salt, rock}
C = {elephant, deer, cow, horse}.

This particular pairing is perhaps a more natural one and is easier to see at a glance than one in which the arrows are crossed. Similarly, in pairing the elements of {blue, song} with those of {National Anthem, color}, it would be "natural" to associate blue with color and song with National Anthem. The fact that the two sets match, however, is exactly as clear if blue is paired with National Anthem and song is paired with color.

Ordering Sets

In pairing the elements of A with those of B, there is a member of B which is not paired with any element of A, and this will be so regardless of how the elements are paired. In this case, we say that B has more members than A. We can also say that A has fewer members than B. Thus, we can compare sets according to three possible outcomes:

A matches B;
A has more members than B;
A has fewer members than B.

Furthermore, all this can be accomplished without counting. Suppose C is the set of all children in the school and S is the set of seats in the school auditorium. By pairing, we can determine without counting whether one set has more members than the other, one set has fewer members than the other, or the sets match.

Problems*

1. Which of the following pairs of sets match? For those that do not match, state which set has more members.
   a. {letters in the word "group"} and {g, o, p, r, u}
   b. {23} and {232}
   c. A = {1, 2, 3, 4, 5} and B = {c, d, e, f}
   d. B = {c, d, e, f} and C = {oyster, walrus, carpenter}
   e. A = {1, 2, 3, 4, 5} and C = {oyster, walrus, carpenter}
2. State why we do not necessarily have a one-to-one correspondence between the addresses in the block and houses on the same block.
3. Show that there is a "natural" correspondence between elements of the following pairs of sets. Which of these correspondences is one-to-one for the pairs of sets?
   a. A = {animal, vegetable, mineral} and
      B = {carrot, plutonium, hippopotamus, beets}
   b. A = {animal, vegetable, mineral} and
      B = {carrot, plutonium, beets}
   c. A = {animal, vegetable, mineral} and
      B = {iron, giraffe, turnip}

In one of the above problems we considered three sets, A, B, C, where

A = {1, 2, 3, 4, 5}
B = {c, d, e, f} and
C = {oyster, walrus, carpenter}.

Note that A has more members than B and that B has more members than C. Moreover, it can be seen that A has more members than C. This illustrates a property that may be fairly obvious. Though obvious, the property is nonetheless important, because it provides us with some means of working with numbers. The property may be stated in general terms as follows.

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* Solutions for problems in this chapter are on page 69.

If A has more members than B
and if B has more members than C,
then A has more members than C.

As before, this property is derived without recourse to counting. The conclusion sanctioned by this property gives us the comparison of A and C with a set, B, acting as intermediary. Thus, this is referred to as a transitive property. (In a sense, it tells us how A compares with C using B as a "yardstick".) Clearly, a transitive property is similarly applicable when A has fewer members than B, and B has fewer members than C. That is,

If A has fewer members than B and
if B has fewer members than C, then
A has fewer members than C.

Furthermore, a similar property holds when the sets match, as we shall show later.

Observe that if A has more members than B, and if C has more members than B, no general conclusion can be made.

For example, if A = {1, 2, 3, 4, 5}, B = {c, d, e}, and C = {oyster, walrus, carpenter, cabbage}, then A has more members than B, C has more members than B, and A has more members than C.

If A = {1, 2, 3, 4, 5}, B = {c, d, e}, and C = {oyster, walrus, carpenter, cabbage, king}, then A has more members than B, C has more members than B, and A matches C.

If A = {1, 2, 3, 4, 5}, B = {c, d, e}, and C = {oyster, walrus, carpenter, cabbage, king, owl, pussy-cat}, then A has more members than B, C has more members than B, and A has fewer members than C

Thus, if A has more members than B, and C has more members than B, it is possible for A to have more members than C, to have fewer members than C, or to match C. So in this case we cannot determine the order of A and C.

By the transitive property, we have a way of ordering sets that do not match. If A = {1, 2, 3}, B = {a, b, c, d, e, f}, and C = {2, 5, b, ★}, then as A has fewer members than C and C has fewer members than B, we might order these sets: A, C, B. If D = {carrots, cabbage, carpenter, carousel, castenet}, by repeated comparison, D would be ordered between C and B, thus: A, C, D, B. Of course, the sets may be ordered equally well by the "more than" relation. For our purpose, ordering by the "fewer than" relation will lead directly to the ordering of numbers according to increasing size.

## Equivalent Sets

One of the possible outcomes from the pairing of the elements of two sets is that the sets match. If each element of A is paired with exactly one element of B and no element of B is left unpaired, we say that A *matches* B. Another way of describing this is that the elements of the sets are in *one-to-one correspondence*. A third way of saying this is that

A is *equivalent* to B.

As we mentioned earlier, the equivalence relation is transitive. If D = {1, 2, 3, 4, 5}, L = {c, d, e, f, g}, and W = {oyster, walrus, carpenter, cabbage, king}, then D is equivalent to L, L is equivalent to W, and D is equivalent to W. We can say this in general for any three sets A, B, and C

(a) if A is equivalent to B and B is equivalent to C, then A is equivalent to C.

If A = {1, 2, 3, 4, 5} and B = {a, b, c}, we can see that A has more members than B. Now if C = {$\alpha$, $\beta$, $\gamma$, $\delta$, $\epsilon$}, then C is equivalent to A; also, C has more members than B. In fact, this notion can be extended to give us a broader picture for comparison.

If A has more members than B, and if C is equivalent to A and D is equivalent to B, then C has more members than D.

A similar statement may be made in connection with the "fewer than" relation. More than that, this concept can be applied to the transitive property. Thus if A has more members than B and B has more members than C, then any three sets that are equivalent to A, B, and C respectively, may be ordered accordingly. In this manner, all sets may be ordered; sets that are equivalent belong in the same place in the order.

For sets that are equivalent, there are two additional properties that are of particular interest to us. If $A = \{1, 2, 3, 4, 5\}$ and $B = \{a, b, c, d, e\}$, then A is equivalent to B. It is equally true that B is equivalent to A. We can see that by our pairing process, it must be true in general that

(b) if A is equivalent to B, then
B is equivalent to A.

This is a property that the "more than" relation does not have. That is to say, if A has more members than B, then it is not true that B has more members than A. Neither does the "fewer than" relation have this property.

If $A = \{1, 2, 3, 4, 5\}$ and $B = \{3, 1, 4, 2, 5\}$, then certainly A is equivalent to B. In fact, here, $A = B$. Recalling that by $A = B$, we mean that both A and B represent the same thing (they are names for the same thing), it is clear that a set is equivalent to itself; that is,

(c) A is equivalent to A.

This is another property that the non-equivalent relations do not have. It is not true that A has more members than A; nor is it true that A has fewer members than A.

On the surface, the statement that a set is equivalent to itself may seem rather trivial. This is another of those statements that will have some repercussions later when we deal with numbers. It is not any more trivial than to assert that

$$2 + 5 = 7 \text{ because } 7 = 7.$$

Moreover, as was pointed out before, the last two properties stated for equivalent sets do distinguish equivalence from non-equivalence. Equivalence relations possess all three of the properties mentioned, which have been identified by the letters (a), (b), and (c).

Problems

4. Write the order of the following sets, beginning with the set that has the fewest members.

   a. A = {letters of the alphabet in the word "peacock"}
      B = {letters of the alphabet in the word "letters"}
      C = {letters of the alphabet in the word "Mississippi"}
      D = {letters of the alphabet in the word "mathematics"}

   b. A = {1, 2, 3, 4}; B = {2, 3, 5, 7, 11, 13};
      C = {a, b, c, d, e, f}; D = { }.

5. Show how the transitive property may be applied to the following sets.

   A = {lion, tiger, leopard, elephant, mouse, cat}
   B = {house, tree, salt, rock}
   C = {the days of the week}

6. If A, B, C are the sets defined in Problem 5, and D, E, F are the sets so that

   D is equivalent to A
   E is equivalent to B
   F is equivalent to C,

   what is the order of D, E, F?

7. If A has more members than B, and C has more members than A, which set has the most members?

8. If B has more members than A, and C has more members than A, which set has the least members?

Applications to Teaching

By the pairings that we have stated above, one member of a set is paired with exactly one member of a second set. Thus, it may be possible that we cannot completely pair the members. If A has fewer members than B, there will be at least one member of B that will left unpaired. Furthermore, in a pairing, no more than one element of A is paired with a particular element of B. So, if A has more elements than B, there will be at least one element of A that will be left unpaired. If no element of either A or B is left unpaired, then we have a one-to-one correspondence (abbreviated: 1-1 correspondence).

There are many many-to-one correspondences. For example, associating numbers with their squares is a many-to-one correspondence. Both -2 and 2 are associated with (or correspond to) 4; both -3 and 3 correspond to 9. The correspondence is 2-to-1 with the exception of the number 0. 0 corresponds to 0. As another illustration, the rounding-off process is many-to-one. In rounding-off whole numbers to the nearest tens, the numbers

35, 36, 37, 38, 39, 40, 41, 42, 43, 44

are either "rounded-up" or "rounded-down" to 40. So this is a ten-to-one correspondence.

Our main concern here is with 1-1 correspondence to arrive at comparing sets according to how many elements they have. This in turn gives us a basis for comparing numbers later on. Aside from this, there will be many occasions in the mathematical career of the students, leading for one thing, to the concept of one-to-one functions that the students will encounter.

The students have been accustomed to thinking about pairs of objects that are alike, for example, pairs of kittens, pairs of shoes, pairs of socks, and so on. In our examples we have avoided the use of the word "pair" in this context because we do not want this restriction to get in the way of the concepts associated with numbers and with counting. Our concentration on set-comparison by equivalence is to prepare for the concept that if sets are equivalent, then they generate for us the same number. It is not the other way around: that if we have the same number of elements, then the sets are equivalent. This is so because we start with pre-number concepts that do not require the knowledge of numbers. It is from such concepts that we derive the concept of numbers. For this reason, we want to avoid the mention of numbers in this development.

The word "equivalent" may cause difficulty for some children. However, this may be again a matter of individual reaction. Some teachers have found that some children apparently cope with this word successfully because the word is foreign to the children's vocabulary. The phrase "as many as" is also used in conjunction with developing the notion of equivalence. The words in this phrase are more easily handled, but the longer phrase demands more attention on the part of the children. Some children may attend to only part of the phrase. For example, in response to the request to produce a set with as many members as a given set, the child may merely produce one with many members.

The notion of separating objects into equivalent sets or classes also underlies our thinking of many names for a number. For example,

$$\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}, \ldots$$

all name the same number, and we can think of all these fractions as being in the same equivalence class. Any fraction in this class is equivalent to another and we may use any one fraction in this set as a representative of the set. Usually, we choose the fraction that is reduced to lowest terms as the representative and consider that this represents the number. But this is not always the case. For example, if we have the problem

$$\frac{1}{2} + \frac{2}{3}$$

neither the fraction $\frac{1}{2}$ nor $\frac{2}{3}$ are convenient representatives for the numbers that we have in mind. From the set of fractions for one-half

$$\{\frac{1}{2}, \frac{2}{4}, \frac{3}{6}, \frac{4}{8}, \frac{5}{10}, \ldots\}$$

and the set of fractions for two-thirds

$$\{\frac{2}{3}, \frac{4}{6}, \frac{6}{9}, \frac{8}{12}, \frac{10}{15}, \ldots\}$$

we pick the convenient ones with common denominators to work with in our problem. Thus

$$\frac{1}{2} + \frac{2}{3} = \frac{3}{6} + \frac{4}{6} \qquad \text{or} \qquad \frac{1}{2} + \frac{2}{3} = \frac{6}{12} + \frac{8}{12}$$

and so on. Out of these, the ones we consider to be the most convenient ones to use are the ones with the least common denominators.

In this example of equivalence classes, the three properties we noted for equivalence relations show up quite clearly. For example,

a. if $\frac{1}{2} = \frac{2}{4}$ and $\frac{2}{4} = \frac{3}{6}$, then $\frac{1}{2} = \frac{3}{6}$;

b. if $\frac{1}{2} = \frac{2}{4}$ then $\frac{2}{4} = \frac{1}{2}$;

c. $\frac{1}{2} = \frac{1}{2}$.

Similarly, underlying the concept of numbers is the concept of equivalent sets. All the sets that are equivalent have the same number of members and we may use any one of these equivalent sets to work with as may be needed. In the next chapter, when we discuss the standard set, we shall indicate how such a set may be used as a representative of the equivalence class. The concept of a number then is the property shared by all members of the equivalence class. In this case, each member of the class (set) is a set.

## Exercises - Chapter 2

1. If the sets match, show a pairing. If they do not, tell which set has fewer members than the other.

   a. A = {□, O, △, ★}
      B = {X, I, V, M, C}

   b. C = {cow, tree, blimp}
      D = {dirigible, trunk, milk}

2. Order the sets X, Y, Z.
   X = {1, 2}   Y = {3, 4, 5, 6}   Z = {789}

3. Gloria is taller than Andrea, and Mary is taller than Gloria. Can the concept of transivity be applied here? If not, why not? If so, what conclusion can be drawn?

4. In attempting to place the elements of P in 1-1 correspondence with the elements of Q, if we run out of members of P before we run out of elements of Q, what can be said of the relationship between P and Q?

5. The elements of which sets can be put in a 1-1 correspondence?

   a. A = {living human beings} B = {functioning human brains}

   b. C = {social security numbers} D = {income tax returns filed}

   c. E = {consonants in "I"} F = {women who have been president of the U. S.}

   d. G = {the human senses} H = {normal number of toes on a dog's hind foot}

6. Name three ways of describing the fact that A matches B.

---

Solutions for Problems

1. a. These sets match. In fact they are equal, so one "natural" pairing would be to pair each member with itself.

   b. These sets match. There is only one pairing since each is a single member set:

$$\begin{tikzcd}\{23\} \arrow[d, leftrightarrow] \\ \{232\}\end{tikzcd}$$

   c. A and B do not match.

$$\begin{tikzcd}[column sep=small] A = \{ & 1, \arrow[d, leftrightarrow] & 2, \arrow[d, leftrightarrow] & 3, \arrow[d, leftrightarrow] & 4, \arrow[d, leftrightarrow] & 5\} \\ B = \{ & c, & d, & e, & f\} & \end{tikzcd}$$

   Since there is an element of A left over in any pairing, A has more members than B.

   d. B has more members than C.

   e. A has more members than C.

2. Many houses are split into two or more living units such as duplexes, each with its own address. There are often more addresses than houses on a block. Also, a house on the corner may be facing another street; in that case, the address is not "in the block". This will depend on the interpretation of "addresses in the block".

3. a.

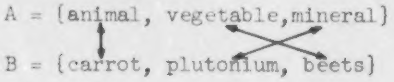
$$\begin{tikzcd}[column sep=small] A = \{ & \text{animal,} \arrow[drr, leftrightarrow] & \text{vegetable,} \arrow[dl, leftrightarrow] \arrow[drr, leftrightarrow] & \text{mineral} \arrow[dl, leftrightarrow] & \} & \\ B = \{ & \text{carrot,} & \text{plutonium,} & \text{hippopotamus,} & \text{beets}\} & \end{tikzcd}$$

   This is not a one-to-one correspondence since vegetable is paired with carrot and beet.

   b.

$$\begin{tikzcd}[column sep=small] A = \{ & \text{animal,} & \text{vegetable,} \arrow[dl, leftrightarrow] \arrow[dr, leftrightarrow] & \text{mineral}\} \arrow[dl, leftrightarrow] \\ B = \{ & \text{carrot,} & \text{plutonium,} & \text{beets}\} \end{tikzcd}$$

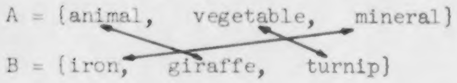
   The "natural" correspondence is not one-to-one since carrot and beets pair naturally with vegetable, and there is no animal in B to pair naturally with animal. However, the sets can be put into one-to-one correspondence as is shown by the possible pairing:



A = {animal, vegetable, mineral}

B = {carrot, plutonium, beets}

c.



A = {animal, vegetable, mineral}

B = {iron, giraffe, turnip}

This natural pairing also demonstrates the one-to-one correspondence between A and B.

4. a. A = {p, e, a, c, o, k}
   B = {l, e, t, r, s}
   C = {m, i, s, p}
   D = {m, a, t, h, e, i, c, s}
   The order requested is C, B, A, D

   b. D, A, B or D, A, C. Since B and C are equivalent sets, they must occupy the same position in any ordering.

5. Since B has fewer members than A and A has fewer members than C, it must be true that B has fewer members than C.
   or
   C has more members than A; A has more members than B. Therefore, C has more members than B.

6. The increasing order of A, B, C is B, A, C; D, E and F must be ordered E, D, F because equivalent sets must occupy the same position in any ordering.

7. C. By transitivity, we can order the sets as B, A, C.

8. A. This is a case in which we can not determine the order of B and C. We only know they both have more members than A.

# CHAPTER 3

## WHOLE NUMBERS

### Number Property of Sets

It was stated in Chapter 1 that from the idea of sets the concept of number would be developed. The discussion in Chapter 2 concerned the ordering of sets. Sets were compared on the basis of characteristics they might have in common. Attention will now be focused on one of these common properties in order to arrive at the number concept.

Recall that sets can be compared according to different criteria. A set of red balloons and a set of red blocks share the common characteristic of color. A set of blue blocks, a set of green blocks and a set of red blocks are each composed of elements which are blocks.

In the last chapter, attention was given to pairing the elements of two sets. If the sets matched, they were said to be equivalent. For example, {Leon, Rosa, Eddy} is equivalent to {a, b, c} because their members can be paired with none left over. It is certainly possible to name many other sets which are equivalent to these; indeed, we could never exhaust all the possibilities. These sets share a common property; that is that they have the same number of members. Similarly, the set {★, ▯ , • , ▬ , □} has this number property in common with all sets equivalent to it because they all have five members.

Every set has this property. We call this characteristic the *number* of the set, and it is determined by the number of elements in the set. Sets which are equivalent have the same number. To simplify the terminology, we denote the number property of a set A as N(A). We can rephrase the statement that equivalent sets have the same number by saying:

> If the sets A and B are equivalent,
> then $N(A) = N(B)$.

Note that this does not say $A = B$. The statement $A = B$ is only true if A and B have the *same* members.

## Problems*

1. Describe a property which the following two sets have in common with each other.

   D = {doll, balloon, tinker toy}
   W = {block, wagon}

2. Identify the number of the sets using the notation N( ).

   a. S = {b, d, f, h, j}
   b. P = {?, ., :, "}
   c. A = {letters in "abbreviation"}

## Order of Numbers

Looking again at the order of sets armed with the concept of number, it is now possible to order numbers. In sets which do not match, it has been said that one set has fewer members than the other. If D = {Tuesday, Thursday} and V = {a, e, i, o, u}, then D has fewer members than V. The number property of D is 2 and the number property of V is 5. Along with the concept that D has fewer members than V, we have the concept that 2 _is less than_ 5. In this manner, the term "has fewer members than" as used with reference to sets, translates very naturally into "is less than" when speaking of numbers. The symbol for "is less than" is written "<". Thus,

D has fewer members than V

implies that

$$N(D) < N(V).$$

Consider the following sets

L = {a, b, c, d, e, f},    F = { ★ , □ , •, ∞ , ▯ },
G = { $\alpha$ , $\beta$ , $\gamma$ }    W = {lion, tiger, celery, fire},
D = {1, 2},    P = {Rosa}.

By repeated use of set comparison, these sets might be ordered thus:

P, D, G, W, F, L.

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* Solutions for problems in this chapter are on page 84.

Since N(P) = 1, N(D) = 2, N(G) = 3, N(W) = 4, N(F) = 5, and N(L) = 6, the ordering of L, F, G, W, D, P as P, D, G, W, F, L gives the ordering of the numbers 1, 2, 3, 4, 5, 6. Where does 9 belong in this order? By repeated comparison, a set with 1 member has fewer members than a set with 9 members. A set with 2 members has fewer members than a set with 9 members. So do sets with 3, 4, 5, and 6 members; each of these has fewer members than a set with 9 members. Thus we have

$$1 < 2 < 3 < 4 < 5 < 6 < 9.$$

In a similar manner, we can locate 8 as being between 6 and 9. A set with 8 members has more members than one with 6 members and has fewer members than one with 9 members. Thus,

$$1 < 2 < 3 < 4 < 5 < 6 < 8 < 9$$

and so on.

Of course the sets may have been ordered by the relation "has more members than." Then the order would be the reverse of what we have here. Associated with the relation "has more members than" for sets, is the relation "is greater than" for numbers. The expression "is greater than" is denoted by the symbol ">". Thus,

$9 > 8$ is read "9 is greater than 8",

and numbers can be ordered accordingly. For the numbers 1, 2, 3, 4, 5, we have the decreasing order.

$$5 > 4 > 3 > 2 > 1.$$

While it is true that the numbers can be ordered by either of the relations, the increasing order (by the $<$ relation) is usually given. By this ordering, the numbers 1, 2, 3, and so on may be progressively ordered one after another beginning with 1 as the least number.

Observe that these numbers that we have so far inherit their properties from sets of objects. In this manner, sets serve as models in the determination of properties of such numbers. For example, the model sets L, F, G, W, D, P that we have specified above help to determine the order of the numbers 6, 5, 3, 4, 2, 1. Any set that is equivalent to L will do as a model for 6; indeed, any of the sets may be replaced by an equivalent set.

If $A = \{a, b, c, d, e, f\}$ and $B = \{\alpha, \beta, \gamma, \delta\}$, then clearly $N(A) = 6$ and $N(B) = 4$. By set comparison, we have $N(B) < N(A)$. One of the sets that B is equivalent to is $C = \{a, b, d, f\}$; there is a 1-1 correspondence between the elements of B and C. Notice that C is a proper subset of A. What we have then is that

> if B is equivalent to a proper subset of A, then $N(B) < N(A)$.

This is certainly true for the kinds of numbers we have been working with. Similarly, it can be seen that

> if A is equivalent to a proper subset of B, then $N(B) > N(A)$.

Recall that in comparing any two sets, A and B, it must be the case that A has more members than B, or A has fewer members than B, or A is equivalent to B. Correspondingly, for numbers, N(A) and N(B), exactly one of the following statements must be true:

$$N(A) > N(B), \quad N(A) < N(B), \text{ or } N(A) = N(B).$$

Problems

3. If $S = \{b, d, f, h, j\}$, $P = \{?, ., :, "\}$, and $A = \{a, b, r, e, v, i, t, o, n\}$, put the sets in increasing order and then order their numbers, using the symbol $<$.
4. Without knowing the numbers of two sets, say X and Y, what must be true of N(X) and N(Y)?

Ordered Sets

Up to now, there has been a good deal of emphasis on the fact that the order in which the elements of a set are listed is immaterial. We have pointed out that sets such as {q, u, i, c, k} and {c, i, k, q, u} are exactly the same. They are simply different ways of listing the elements of {letters in the word "quick"}. Similarly, {3, 2, 1}, {2, 1, 3}, and {3, 1, 2} are equally correct ways of describing the set, {1, 2, 3}.

On the other hand, it is certainly true that it is natural and convenient to name many sets in a particular order. For instance, most English speaking people would list the members of the set of vowels as {a, e, i, o, u}. It is natural because they were learned in that order. It is convenient because without undue checking one can be sure he has not omitted any member. However, in this case, it would not cause much difficulty to write the set as {i, u, e, o, a}. A more striking example which supports the convenience of a particular order can be found in the alphabet. It would be very difficult to list the alphabet in any order other than {a, b, c, d, e, f, g, h, i, j, k, l, m, n, o, p, q, r, s, t, u, v, w, x, y, z}. Try to say it backwards! Any other attempt to list the letters, whether systematically or randomly, would very likely result in some omissions. Nevertheless, any listing which contains all twenty-six letters is the same set as {a, b, c, ... , x, y, z}. As in ordinary writing, the use of the ellipsis "..." in the notation {a, b, c, ..., x, y, z} indicates the omission of certain words or elements. In its use is implied an understanding as to what elements are omitted in the writing. Certainly, the order in which the elements have been listed here communicates which elements are omitted.

## Number Names

We could cite numerous examples in which order is apparent. The concept of order is a brilliant characteristic of our number system. This suggests perhaps the best examples of naturally ordered sets. If N = {one, two, three, four, five, ...}, the elements are ordered in the listing. Some subsets of N are

{one, three, four, six}, {two, one, four, five, three},
{three, four, five, six}.

Of these subsets of N, the first and the third have their elements listed in an order that is associated with set comparison. There is a special set of such ordered subsets of N that will be particularly useful to us. These are the subsets

{one}, {one, two}, {one, two, three}, {one, two, three, four}, ...

We can see that the number property of {one} is 1, the number property of {one, two} is 2, the number property of {one, two, three} is 3, and so on. A distinct feature of these ordered sets then, is that the last element listed names the number property of the set.

Because of this feature, we find it convenient to use these as model sets for numbers. To illustrate the application, suppose S = {★, balloon, doll, ☽, key}. By the pairings,

$$\begin{tikzcd}[column sep=small, row sep=small]
S = \{ & ★, \arrow[d, leftrightarrow] & \text{balloon}, \arrow[d, leftrightarrow] & \text{doll}, \arrow[d, leftrightarrow] & ☽, \arrow[d, leftrightarrow] & \text{key}\} \arrow[d, leftrightarrow] \\
F = \{ & \text{one}, & \text{two}, & \text{three}, & \text{four}, & \text{five}\},
\end{tikzcd}$$

we can see that S is equivalent to F. Since N(F) = 5, the number property of S must be 5. In essence then, these model sets

{one}, {one, two}, {one, two, three}, {one, two, three, four}, ...

which we call the "standard sets" help us to arrive at the number property. Moreover, we arrive at the number property very conveniently by these standard sets because the last element of each names the number. Here we see too, that the numbers 1, 2, 3, 4, ... are the numbers we associate with counting. Thus the ordered set,

{1, 2, 3, 4, 5, 6, ...}

is known as the set of counting numbers. This set is also referred to as the set of natural numbers.

The implication for the use of the standard sets is that whenever we want the number property of a set, we try to match this set with one of the standard sets. When we find the standard set that is equivalent to our set, we get the number property immediately. In practice, the process is more simple. Implicitly, we start with the set of number names

N = {one, two, three, four, five, six, ...}

and we identify the standard set by pairing. Thus, the pairings

$$\begin{tikzcd}[column sep=small, row sep=small]
N = \{ & \text{one}, \arrow[d, leftrightarrow] & \text{two}, \arrow[d] & \text{three}, \arrow[d] & \text{four}, \arrow[d] & \text{five}, \arrow[d] & \text{six}, \ldots\} \\
S = \{ & ★, & \text{balloon}, & \text{doll}, & ☽, & \text{key}\} &
\end{tikzcd}$$

identify F = {one, two, three, four, five} as the set that S is equivalent to. Of course, we normally by-pass the construction of F since our last pairing gives us the information we need. Nonetheless, the pairing with elements of N is precisely the process we use with counting; each object that is counted is associated with a number.

Problems

5. For each of the following sets, state whether the elements are apparently ordered; if an order is apparent describe what might be the determination of the order.

   a. {1, 2, 3, 4, 5}
   b. {5, 4, 3, 2, 1}
   c. {accordion, albatross, brain, bubble, gum, humbug}
   d. {student, teacher, principal, superintendent}
   e. {father, son, mother, daughter}
   f. {father, mother, son, daughter}
   g. {q, w, e, r, t, y, u, i, o, p}
   h. {5, 4, 1, 3, 2}

6. Given an ordered set,

   H = {thumb, index, middle, ring, pinky},

   a. Show the 1-1 correspondence between the elements of H and a standard set.
   b. If S = {Dorothy, Rosie, Laurie, Nancy, Susan} give a subset of H that is equivalent to S; what is N(S)?
   c. Describe how counting on our fingers implies finding a set that is equivalent to a standard set of number names.

It is now possible to identify the number of all sets that have members. The set without members has been neglected for obvious reasons. None of the standard sets is equivalent to the empty set. In fact, the only set that is equivalent to the empty set is { }; there is only one set that has no members. The number property of the set is 0 and the number name is "zero." Since each of the standard sets of number names has at least one member that is not a member of { }, and since { } is a subset of each set, the empty set is a proper subset of each. By the statement that

if B is equivalent to a proper subset
of A, then $N(B) < N(A)$,

we can determine the location of 0 in the order 1, 2, 3, 4, ...; it precedes every other number.

Thus, we now have the ordered set,

$$W = \{0, 1, 2, 3, 4, \ldots\}.$$

We call W the set of whole numbers. Note that unless we are mighty careful, if we use

{zero, one, two, three, four, five, six, ...}

to arrive at the number of elements, we may be short of the count. Therefore, we do not consider W as the set of counting numbers; the set of counting numbers is the proper subset of W that has all the elements of W excepting 0.

With the inclusion of { } as one of the standard sets, we can now say that for any set there is an equivalent standard set from which the name of the number property is determined. The standard sets used to derive the number property of any set are then,

{ }, {one}, {one, two}, {one, two, three}, ...

With all the standard sets other than { }, identification of the number property of any set is immediate. The last element names the number.

A natural suggestion that might be offered is to substitute {zero} for the empty set in the above list so that we can get the names immediately from the standard sets without exception. To see why this cannot be done, we can consider the meaning of the notation {zero}. This denotes "the set whose member is "zero"." Hence, there is a member in {zero}; the member is "zero". In contrast, { } has no member. Recall that the process in finding the number property of a given set involves finding the standard set that is equivalent to the given set. {zero} is not equivalent to { }. For the same reason that {zero} is not to be confused with { }, the set specified as {0} should not be confused with the empty set. There is one member in {0}; the member is "0"; the empty set does not have this element as a member.

## Cardinality and Ordinality

There are two aspects of number which have been inherent in this chapter. The conception of number as a quantity, as in how many objects, is called the cardinality of a number. The number property of a set is a cardinal number. The words "one", "eight", "ninety-nine", and so on, are names of cardinal numbers. This concept can be considered entirely separately from the phenomenon of order.

Much has been said about the ordering of sets and of elements within sets. In this reference the words first and last have been used. The fact that we can talk about the third letter of the alphabet or the fiftieth state of the Union, depends on the ordinality of numbers. The words first, second, thirty-eighth and so on are names of ordinal numbers. These are independent of quantity and can only be considered relative to some frame of reference. That is, we cannot speak of the third quarter in a football game without inferring that there were a first and a second quarter. However, the third quarter only refers to one of the inferred three quarters. Both aspects of number are contained in the statement: Jimmy is the third child of our seven children. Note that an ordinal number requires a set of at least the corresponding cardinal number of members. Jimmy is the third child requires at least a set of three. On the other hand, a cardinal number does not necessitate ordinality of its members. The number two is the property of {chicken, egg}; the question of the ordinality of this set has occupied minds for years!

At this point, we want to remark on the common usage of language with reference to the ordinality and cardinality of numbers. Quite often, as in the case of "Page 3", a number is meant to be used in an ordinal sense even though it is stated as a cardinal number. The identification "Page 3" refers to the third of a series of pages rather than to three pages.

## Problems

7. Identify each number in the following statements as to whether the use is ordinal or cardinal.

"Two of you in Group Three will have the assignment of looking up the history of zero in the encyclopedia. You will find that of all these 30 volumes, Volume 17, Part 2 contains material that will be most helpful to either one of you."

## Finite and Infinite Sets

As in the case of the notation {a, b, c, ..., x, y, z}, we have used the ellipsis in 0, 1, 2, 3, 4, ... to indicate the omission in writing of certain elements. The difference in the latter use of the three dots is that no end is indicated in the list of whole numbers. Of course this is the intention since the whole numbers form an infinite set. As implied by the term, the elements in an infinite set are inexhaustible.

It has not been suitable in the treatment of sets so far to consider the possibility of infinite sets. Many of the properties developed do not hold true for infinite sets. Rather than prefacing all ideas with a statement about exceptions, we have simply overlooked the subject. The intention now is to mention infinite sets, to consider certain implications to what is known, but not to go into all ramifications to any great depth.

Infinite sets are permitted within the scope of the definition of a set. In the example of

$$W = \{0, 1, 2, 3, 4, 5, \ldots\}$$

it is clear what objects are to be included as elements of the set. Another infinite set is that composed of all fractional numbers. Both of these examples of infinite sets can be verified as actually satisfying the requirements of being a set. We can answer the question, "Is this number a member of the set?" for any number we choose.

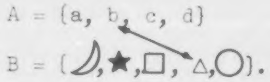
A set is finite if there is a whole number that will answer the question, "How many elements are there in the set?" It is tempting to think that the set of all grains of sand on a particular beach is an infinite set. It may be difficult, in fact impossible in practice, to count them, but at any given instant there are a finite number, albeit very large.

Since numbers offer a wide variety of infinite sets, these sets are certainly significant in arithmetic. It is not possible to consider the number of an infinite set, however.

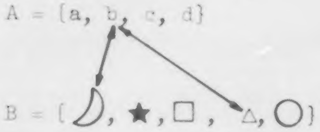
## Applications to Teaching

The focus on sets and other pre-number concepts provides a background for the concept of number introduced in this chapter. If there is a 1-1 correspondence between the elements of two sets, then they are said to be equivalent and have the same number property. To determine whether there is a 1-1 correspondence between the elements of two sets, an element of one set is paired with an element of the other set. The use of the word pair is non-mathematical. The two elements that are associated form a pair.

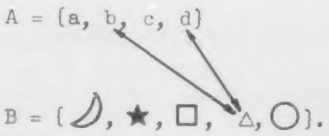
For our purpose, once a pair is so determined, neither of the elements in the pair is to be associated with any other element to form another pair. Thus, for A = {a, b, c, d} and B = {☽, ★, □, △, ○}, if we decide to pair b with △, we have



If, in the attempt to get a 1-1 correspondence, b is paired with △ then b is not to be paired with any other member of B. Neither is △ to be paired with any other member of A. Thus we cannot seek a 1-1 correspondence between the members of A and B by



or



Of course, b may have been selected to be paired with ☽ at the outset. Then b is not to be paired with ★, □, △, or ○; nor is ☽ to be paired with a, c, or d. The one-to-many and many-to-one correspondences illustrated in these last two diagrams as well as many-to-many correspondences will be discussed in the next chapter.

In the children's books the pairings are indicated by connecting lines from one object to another much as we have used the arrows on these pages. Special attention may need to be devoted to explaining the meaning of these extra lines (dotted or otherwise) on the printed page as they may be a source of confusion. Having the children trace over these lines themselves may be helpful. Thus they are actively engaged in connecting or pairing the objects. For the same reason, active participation in pairing objects on the flannel board using yarn to define the pairings will be helpful. The pairings may result in exhausting the elements of one set without exhausting the elements of the other. When the elements of both sets are exhausted simultaneously, the sets match.

Pairing then refers to the elements and matching to the sets. When the sets match, they are said to be equivalent. Thus equivalence implies the same number of elements. Note that equal sets are always equivalent. Sets are equal only if they have the same members; therefore, the number of elements must be the same. However, it is not true that equivalent sets must necessarily be equal. $\{1, 2, 3, 4, 5\}$ and $\{1, 2, 3, 4, 6\}$ are equivalent but not equal; there is a 1-1 correspondence between the elements of these two sets.

Just as the determination as to whether the number properties of two sets are equal depends on whether the sets match, the order of two numbers depends on the result of set comparison. If $A$ is equivalent to a proper subset of $B$, then $N(A) < N(B)$. The characteristics ascribed to numbers derive from characteristics observed for sets, not the other way around.

Properties such as the one stated above concerning the equivalence of a set to a proper subset of another set refer to finite sets. These are not appropriate statements for infinite sets. In fact, the student will encounter later that a set is defined to be infinite if there is a 1-1 correspondence between the elements of the set and elements of a proper subset. At the elementary level, we approach the concept from the more intuitive point of view that the elements are inexhaustible. In the definition of a finite set the criterion that we give is: If there is a whole number that states how many elements there are in the set, then the set is finite. Accordingly, if no whole number may be associated as the number property, then the set is infinite.

Occasionally, a child may understand 1-1 correspondence and the process of counting but still may be unsuccessful because he cannot keep track of what he has counted and what he has not counted. For such a child it may be necessary to actually suggest some systematic strategies in attacking the problem. For example, if the objects are in a horizontal row and he still skips around counting the objects in a random fashion, it might be suggested that he proceed from left to right as in reading.

In counting, it does not matter which element of a set is paired with a given element in the appropriate standard set. The same number property is obtained regardless of the pairings used. By contrast, in ordinal use of numbers, it is assumed that there is a pre-determined order in the given set as well as in the standard set. That is, the elements are ordered by associating with each element as the first, second, third element and so on as the case may be. The ordinal numbers may not be in the vocabulary of some children. However, it has been observed that many children do know what these words mean. In such cases, apparently some incidental learning has occurred.

## Exercises - Chapter 3

1. Name the number property of the sets:

   a. {□, △, ○, ⬡}
   b. {243}
   c. {zero}
   d. {letters in the word "deeded"}
   e. {the number of vowels in "bureau"}
   f. {counting numbers less than 1}

2. Here are four sets: A = {a, b, c, d}
   B = {1, 2, 3}
   C = { }
   D = {$\alpha$, $\beta$, $\gamma$}

   Identify the number properties of these sets; write all the relationships you can, using the numbers and the symbols <, =, >. For example, $3 < 4$.

3. Suppose you want to explain "wide" to someone who speaks no English and you do not speak his language. How would you go about conveying to him the idea of "wide"?

4. M = {man, fish, ape, amoeba, lizard}
   Rewrite the elements of M in some more intuitively logical order and describe how it is determined.

5. List the elements of {9, 6, 11, 4, 3, 1, 10, 8, 7, 5, 2} in such a way that its number can be determined without counting.

6. Which sets are finite and which infinite? If finite, state the number.

   a. {0, 1, 2, 3, 4, ..., 9}
   b. {whole numbers greater than a million}
   c. {1, 2, 3, ...}
   d. {natural numbers less than 16}

7. Distinguish whether the number on each statement is used as a cardinal or an ordinal number; briefly say why.

   a. "Whole Numbers" is the title of Chapter 3 of this book.
   b. There are fifty states in the United States.
   c. The Normans conquered England in 1066.

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### Solutions for Problems

1. Answers may vary; e.g., they are both sets of objects which are children's toys.

2. a. $N(S) = 5$
   b. $N(P) = 4$
   c. Since $A = \{a, b, r, e, v, i, t, o, n\}$, $N(A) = 9$.

3. The sets in increasing order are P, S, A. Since $N(S) = 5$, $N(P) = 4$ and $N(A) = 9$, the numbers must be ordered $4 < 5 < 9$.

4. Exactly one of the following statements must be true: $N(X) < N(Y)$ or $N(X) = N(Y)$ or $N(X) > N(Y)$.

5. a. The elements are listed in increasing order.
   b. The elements are listed in decreasing order.
   c. These words are in alphabetical order.
   d. This describes the positions in the ascending hierarchy in a school.
   e. Males are listed first in decreasing order of age and then females are listed in order of age.
   f. Two apparent rules of order operate here also. The adults are listed before the children and male takes precedence over female.
   g. No order is obvious, so "not apparent" would be correct. However, the letters happen to be in the order in which they appear on the third row of a standard typewriter.

h. Again, no order is apparent. If the set is renamed by the words {five, four, one, three, two}, you can see they are in alphabetical order.

6. a. H = {thumb, index, middle, ring, pinky}
↕ ↕ ↕ ↕ ↕
{one, two, three, four, five}

b. The subset of H which is equivalent to S is the improper subset H = {thumb, index, middle, ring, pinky}. N(S) = 5 as part a. shows.

c. As one counts, he is listing the elements of a standard set. When counting on our fingers, we usually touch one and say a number. This is pairing fingers with numbers so that the sets match. Hence we have found a set, a set of fingers, which is equivalent to {one, two, three}.

7. Cardinal numbers are: "two", "zero", "30", "one"; those used as ordinal numbers are: "Three", "17", "2".

## CHAPTER 4

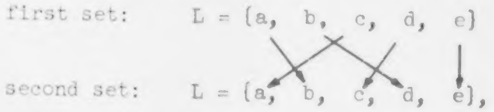
## SET OPERATIONS

### Meaning of an Operation

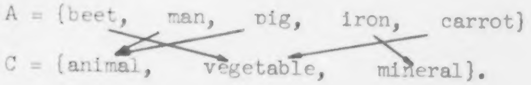
In assigning the number property to a set, we associate with each element of the given set exactly one element of a particular standard set. In comparing two sets, the same process is involved. Both of these are instances of an operation:

> an element of one set is associated with
> exactly one element of a second set.

Of course, as in pairing elements to show that a set is equivalent to itself, it is entirely possible for the second set to be the same as the first. Thus, in the correspondences


first set: L = {a, b, c, d, e}
second set: L = {a, b, c, d, e},

an operation is in effect. Equally, we have an operation in the correspondences


A = {beet, man, pig, iron, carrot}
C = {animal, vegetable, mineral}.

Unlike the notation used in pairings, a single-headed arrow is used here to indicate that with an element in the first set is associated the one element in the second set. The double-headed arrow used in a pairing indicates that with an element in the first set is associated the one element in the second set; furthermore, with the one element in the second set is associated the one element in the first set.

Note that even though more than one member of A may be associated with a member of C (for example, man and pig are both associated with animal), we have an operation. Attached to each element of A is exactly one element of C.

Included in the concept of an operation are not only the two sets that are involved in the correspondences; of equal importance is the rule that tells us how each correspondence is established. For example, in the illustration of an operation from L to L that we have above, the rule is simply given by the arrows. This "rule" states that a is to correspond with b, b is to correspond with d, c is to correspond with a, and so on. In the illustration of an operation from A to C, again the rule may be considered to be defined by the arrows. Thus,

> beet and carrot both correspond to vegetable,
> man and pig both correspond to animal, and
> iron corresponds to mineral.

What may be more in evidence is that the rule here might also be appropriately described as: "the classificiation of the elements of A as to one of three categories ..."

Apart from these constituents of an operation, namely two sets and a rule, the requirement that exactly one member of the second set is to be associated with a member of the first set must be emphasized. This requirement is precisely what is meant by the assertion that an operation is uniquely defined:

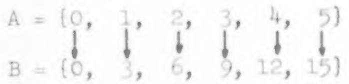
> One and only one (unique) member of the second set corresponds to a member of the first.

If a member of the first set is associated with no element of the second set, then the operation is not defined for this particular member. However, it may be that the operation is still defined for other members of the first set.

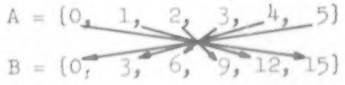
Problems *

1. For each of the following correspondences, state whether an element of the first set is assigned a unique element of the second set; if not, give a reason.

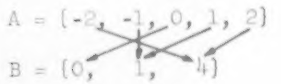
a.



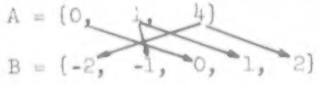
b.



c.



d.



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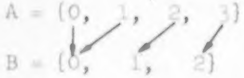
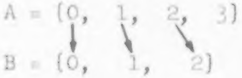
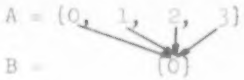
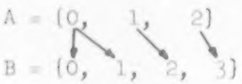
* Solutions to problems in this chapter are on page 103.

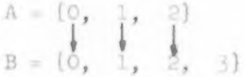
2. State whether each of the following define an operation; if not, explain why not.

a.


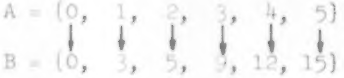
b.


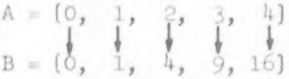
c.


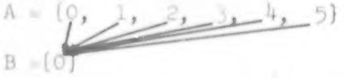
d.


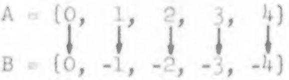
e.


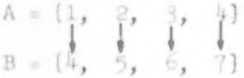
3. If possible, give a verbal description of the rule for each of the following operations.

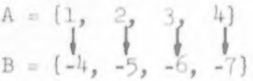
a.

A = {0, 1, 2, 3, 4, 5}
B = {0, 3, 5, 9, 12, 15}

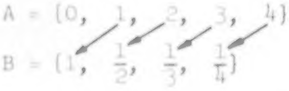
b.

A = {0, 1, 2, 3, 4}
B = {0, 1, 4, 9, 16}

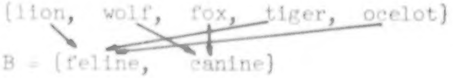
c.


d.

A = {0, 1, 2, 3, 4}
B = {0, -1, -2, -3, -4}

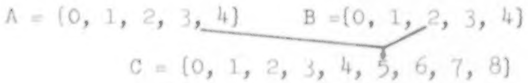
e.

A = {1, 2, 3, 4}
B = {4, 5, 6, 7}

f.

A = {1, 2, 3, 4}
B = {-4, -5, -6, -7}

g.

A = {0, 1, 2, 3, 4}
B = {1, 1/2, 1/3, 1/4}

h.

{lion, wolf, fox, tiger, ocelot}
B = {feline, canine}

The operations that we have considered so far are the more simple ones. To an element of a first set is attached an element of a second set. The kinds of operations that are more in evidence in mathematics are a little more involved. We usually have three sets to work with. But as before, two or all three of these sets may be identical. For example, if $A = \{0, 1, 2, 3, 4\}$, $B = \{0, 1, 2, 3, 4\}$, and $C = \{0, 1, 2, 3, 4, 5, 6, 7, 8\}$ we have a correspondence defined as follows:


A = {0, 1, 2, 3, 4}
B = {0, 1, 2, 3, 4}
C = {0, 1, 2, 3, 4, 5, 6, 7, 8}

Here, the combination of the element 3 from A and the element 2 from B is associated with the element 5 from C. If this were the operation we normally call addition, then the combination of the element 4 from A and the element 3 from B would be associated with the element 7 from C. Notice that the combination of 2 from A and 5 from B is also associated with the element 7 from C. There should be no implication here that the operation is not unique. The combination of 3 and 4 yields the one number 7. For this operation, 3 and 4 is not associated with any other number in C. Similarly, the combination 2 and 5 yields the one number 7. We shall expand on the topics of the arithmetic shortly. As these operations are directly derived from the operations with sets, we would like to direct our attention at this time to operations with sets.

Set Union

As with the arithmetic operations where the combination of two given numbers yields a third number, with set operations, a third set may evolve from two given sets. One of the set operations that we shall deal with forms this third set by having it composed of all the elements in each of the two sets. This is called the union or join of the two sets, and is denoted by the symbol "$\cup$". For example, if

$$A = \{a, b, c\} \quad \text{and} \quad B = \{d, e, f, g, h\},$$

then the union (join) of A and B, written $A \cup B$, is:

$$\{a, b, c, d, e, f, g, h\}.$$

For our purpose, we use this operation of joining only if the two sets do not have any members in common. The elements a, b, c, are members of A but not of B. It is equally true that none of the members of B is a member of A. If two sets do not have any members in common, as is the case here, then we say that these are disjoint sets. For example, the set of boys in a classroom and the set of girls in the classroom are disjoint sets; the union of these two sets is the set of boys and girls in the classroom.

Problems

4. Find the union of each pair of sets:
    a. $A = \{1\}$; $B = \{2\}$

b. $A = \{1, 2\}$; $B = \{3\}$

c. $A = \{1, 2, 3\}$; $B = \{\ \}$

d. $D = \{1, 2, 3\}$; $L$ ; $\{a, b, c, d, e\}$

e. $L = \{a, b, c, d, e\}$; $D = \{1, 2, 3\}$

f. $S = \{a, b, c, d, e, 1, 2, 3\}$; $G = \{\alpha, \beta, \gamma, \delta\}$

g. $D = \{1, 2, 3\}$; $G = \{\alpha, \beta, \gamma, \delta\}$

h. $L = \{a, b, c, d, e\}$; $A = \{1, 2, 3, \alpha, \beta, \gamma, \delta\}$

5. a. Which of the problems above illustrates the result of the union of a set and the empty set?

   b. If $A$ is a set, what is $A \cup \{\ \}$?

6. If $V$ = {aardvark, bear, cougar, deer, elephant, fox, giraffe, hyena, ibex, jackal, kangaroo, llama},

   $W$ = {aardvark, cougar, fox, jackal},

   and $X$ = {bear, deer, elephant, giraffe, hyena, ibex, kangaroo, llama},

   what is $W \cup X$?

## Properties under Union

There are a few properties under the union operation that will have important bearing for us when we work with numbers. If

$B$ = {Anthony, Barry, Charles, Douglas}

and $G$ = {Ethel, Florence, Grace},

then $B \cup G$ = {Anthony, Barry, Charles, Douglas, Ethel, Florence, Grace}

and $G \cup B$ = {Ethel, Florence, Grace, Anthony, Barry, Charles, Douglas}.

Observe that $G \cup B$ has the same members as $B \cup G$, hence

$$G \cup B = B \cup G.$$

In fact, it is always true that for any two sets it is immaterial whether the first set is joined to the second or the second is joined to the first; the same set results by the union. To express this fact, we say that

sets are commutative under the union operation.

In other words, if $A$ and $B$ are sets, the commutative property under the union operation states that

$$A \cup B = B \cup A.$$

Another way to describe this is: under the union operation, the order of joining does not matter.

There are many actual situations in which commutativity holds. These are instances of commutative operations. For example, it makes no difference in what order the left sock or the right sock is put on. The final result of applying the operation on both objects is identical in each case.

Taking three red marbles from a sack, four green marbles from a second sack, and putting these together into a third sack is another illustration of a commutative operation. Taking four green marbles from the second sack and three red marbles from the first to put together into the third sack would net the same result.

On the other hand, there are situations where the results do depend on the order in which the operation is carried out. For example, applying a coat of red paint on top of a coat of green paint gives a different visual effect than reversing this procedure. Therefore, it is pertinent to point out commutativity when it does occur.

Another property for sets that is of significance for numbers is known as the associative property under the union operation. The combination of more than two sets is involved in this property. Such a combination is possible because a set is specified by the union of two sets with a set so formed; the union with still another set may be obtained. To illustrate, if $D = \{1, 2, 3\}$ and $L = \{a, b, c, d, e\}$, then

$$\begin{aligned} D \cup L &= \{1, 2, 3\} \cup \{a, b, c, d, e\} \\ &= \{1, 2, 3, a, b, c, d, e\}. \end{aligned}$$

If $G = \{\alpha, \beta, \gamma, \delta\}$, we can join $G$ to $D \cup L$ to get

$$\begin{aligned} (D \cup L) \cup G &= \{1, 2, 3, a, b, c, d, e\} \cup \{\alpha, \beta, \gamma, \delta\} \\ &= \{1, 2, 3, a, b, c, d, e, \alpha, \beta, \gamma, \delta\}. \end{aligned}$$

The parentheses around $D \cup L$ shows that $L$ is first joined to $D$. As $D \cup L$ is a set, it is possible to join another set to $D \cup L$, and this is indicated by the union with $G$. Thus new sets may be formed successively by this process. The possibility of creating new sets repeatedly is crucial to the property that we have in mind. However, the property further, and more directly, has to do with examining the result of the repeated joinings.

If D, L, and G are as above, we see that we can get the result of $(D \cup L) \cup G$. Let us now consider the union of L and G, and then join this set to D. The union of L and G is

$$L \cup G = \{a, b, c, d, e\} \cup \{\alpha, \beta, \gamma, \delta\}$$
$$= \{a, b, c, d, e, \alpha, \beta, \gamma, \delta\}.$$

The union of D and this set is then

$$D \cup (L \cup G) = \{1, 2, 3\} \cup \{a, b, c, d, e, \alpha, \beta, \gamma, \delta\}$$
$$= \{1, 2, 3, a, b, c, d, e, \alpha, \beta, \gamma, \delta\}.$$

Compared with

$$(D \cup L) \cup G = \{1, 2, 3, a, b, c, d, e, \alpha, \beta, \gamma, \delta\}$$

that we have above, it is clear that the same set results from the two procedures. In general,

for sets A, B, and C, it is true
that $(A \cup B) \cup C = A \cup (B \cup C)$.

What is conveyed by this property is that B may be joined with either A or C first; the final result of the union of all three sets will be the same. That is to say, B may be associated first with A to form $A \cup B$ or B may be associated first with C to form $B \cup C$; the union of either of these with the remaining set (C or A as the case may be) is the same in both cases. This is what we mean when we say that sets are associative under the union operation. In other words, in a union involving three sets, the different ways the sets are grouped to form unions in the intermediate stage does not affect the final result.

Because we have this option in grouping, $(A \cup B) \cup C$ and $A \cup (B \cup C)$ denote the same set. Consequently, we need not specify in the notation how the union is to be accomplished. Therefore, the notation may be simplified by dropping the parentheses and writing

$$A \cup B \cup C.$$

Since $A \cup B \cup C$ is a well-defined set, the union may be extended again and again. With the same kind of analysis, it is clear that the associative property under the union operation is equally applicable to more than three sets.

A third property that will be of interest to us is one illustrated by the example $\{1, 2, 3\} \cup \{\ \}$. As the union is composed of all the elements in each of the two sets, and since the empty set has no members, the union is precisely $\{1, 2, 3\}$. Therefore, we must have

$$\{1, 2, 3\} \cup \{\ \} = \{1, 2, 3\}.$$

In general, if $A$ is a set, then it is true that

$$A \cup \{\ \} = A.$$

This is parallel to the situation in arithmetic when 0 is involved in addition, such as: $3 + 0 = 3$. In fact, it will be pointed out that the union with the empty set is the basic concept from which we derive the property of 0 under addition.

Finally, another property that we shall be interested in is the one illustrated by our example of the set of animals, aardvark, bear, cougar, and so on. To simplify matters, let us use the initial letter of each for abbreviation. Then we have

$$V = \{a, b, c, d, e, f, g, h, i, j, k, l\},$$

$$W = \{a, c, f, j\} \text{ and } X = \{b, d, e, g, h, i, k, \text{[illegible]}\}$$

A quick inspection will show that $W$ and $X$ are disjoint sets. Moreover, we have found that

$$W \cup X = V.$$

This situation is described by saying that $W$ is the relative complement of $X$, or, in short "$W$ is the complement of $X$". The word, relative, is included in the fuller description to call attention to the fact that, referring to $V$, $W$ is the complement of $X$. What we mean is, the elements in $W$ are the elements of $V$ that are not elements of $X$. By the same token, $X$ is the complement of $W$ relative to $V$.

Many examples of complements abound in actual situations. For example, the set of boys in the classroom is the complement of the set of girls in the classroom. These are disjoint sets that together complete the set of boys and girls in the classroom. The set of vowels and the set of consonants might be another example of complementary sets. A complementary set will also be referred to as a remainder set. This concept will be developed further when we talk about subtraction.

Problems

7. In each of the following problems, state which property or properties are indicated.

   a. $\{a, b, c\} \cup \{d, e, f, g\} = \{d, e, f, g\} \cup \{a, b, c\}$

   b. $A \cup (B \cup C) = A \cup (C \cup B)$

   c. All the children in K-1 or the children in the second grade are children in the kindergarten or children in the first two grades.

8. State which property or properties are indicated in a., b., and c. below:

   Let $A = \{1, 2, 3\}$, $B = \{4, 5\}$, $C = \{6, 7, 8\}$, $D = \{1, 2, 3, 4, 5\}$, and $E = \{4, 5, 6, 7, 8, 9\}$.

   a. $B \cup C = E$ b. $D = A \cup B$

   *c. $A \cup E = D \cup C$

9. If $M$ = {the male presidents of the United States before 1965} and $F$ = {the female presidents of the United States before 1965}, then $M \cup F = M$. State the property indicated.

10. State which of the following activities are commutative.

    a. Go two blocks west and then three blocks north.

    b. Put on the left shoe and then the right shoe.

    c. Put on socks and then shoes.

    d. Open the door and then walk into the room.

    e. Close the hatch and then submerge the submarine.

    f. Put on the hat and then the jacket.

Set Intersection and Properties

In the union, a third set is created from two given sets by pooling together all the elements in each of the two sets. There is another standard way of creating a third set from two given sets. Suppose that one group for reading consists of Charlie, Linus, Lucy, and Snoopy. Suppose also, that one group for mathematics consists of Lucy, Snoopy, Schroeder, Charlotte, and Violet. Then we have two sets,

$R$ = {Charlie, Linus, Lucy, Snoopy}

$M$ = {Lucy, Snoopy, Schroeder, Charlotte, Violet}.

---

*Problems preceded by * are more challenging.

Evidently, these sets are not disjoint: Lucy and Snoopy are members of both. In fact, common members of two sets suggest a natural set of elements--namely, the set consisting of all members that the sets have in common. Associated with two given sets then, is the set whose members are simultaneously elements of both given sets. This set operation is called the intersection of the two sets, and is denoted by the symbol "$\cap$". Thus,

$$R \cap M = \{\text{Lucy, Snoopy}\}.$$

As is the case of the union, the intersection is also uniquely defined. For any two sets, a single set is determined by their intersection.

As $R \cap M$ draws on members of $R$ for its creation, the intersection must necessarily be a subset of $R$. Likewise, $R \cap M$ must necessarily be a subset of $M$. Thus, the intersection is a subset of both sets.

Even when two sets are disjoint, we can specify a set of the common elements; this is simply the set that has no members. Recalling that the empty set is a subset of every set, we see that the statement,

> the empty set is the intersection of two disjoint sets,

is consistent with the observation we have just made, namely, that the intersection must be a subset of each set. Related to this, of course, is that the intersection of a set and its complement is the empty set. This is by virtue of the fact that a set and its complement are disjoint. Another consequence of the fact that the intersection must be a subset of each of its generating sets relates to the intersection of a subset with its super-set. This will be the theme of one of the problems to follow.

Problems

11. For each of the following problems, state whether $C$ represents the union or the intersection.

    a. $A = \{1, 2, 3, 4\}$, $B = \{2, 4, 5, 9\}$, $C = \{2, 4\}$
    b. $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$, $C = \{1, 2, 3, 4, 5, 6, 7\}$
    c. $A = \{1, 2, 3\}$, $B = \{4, 5, 6, 7\}$, $C = \{\ \}$
    d. $A = \{1, 2, 3, 4, 5\}$, $B = \{1, 2, 3\}$, $C = \{1, 2, 3\}$

e. $A = \{1, 2, 3, 4\}$, $B = \{1, 2, 3, 4\}$, $C = \{1, 2, 3, 4\}$

f. $A = \{1, 2, 3, 4\}$, $B = \{\ \}$, $C = \{\ \}$

g. $A = \{1, 2, 3, 4\}$, $B = \{\ \}$, $C = \{1, 2, 3, 4\}$

h. A = {stockholders of Linus Imports, Inc.},
B = {stockholders of Susan Exports, Ltd.}
C = {stockholders of both corporations}

i. A = {stockholders of Linus Imports, Inc.},
B = {stockholders of Susan Exports, Ltd.},
C = {stockholders of the Linus-Susan merger}.

12. Find the intersection of each:

a. $A = \{a, b, h, i, k, o, m, w, z\}$, $B = \{c, e, h, j, o, r, w, x\}$

b. A = {brown-eyed, green-eyed, blue-eyed, raven-haired, brunette, blond, platinum},
B = {pink-eyed, blue-eyed, ox-eyed, black-eyed, red-headed, blond, gray-haired}.

13. If B is a subset of A, what is $A \cap B$?

We have examined properties of sets under the union operation in view of possible applications to numbers. Even more, we shall find that these properties are equally applicable to sets of geometric objects. The same is true about properties of sets under the intersection operation. We have seen that if

R = {Charlie, Linus, Lucy, Snoopy},
and M = {Lucy, Snoopy, Schroeder, Charlotte, Violet},
then $R \cap M$ = {Lucy, Snoopy}.

Note that $M \cap R$ is also {Lucy, Snoopy}. It is obviously a point in logic that if Lucy and Snoopy are members common to R and M, then these same elements are members that are common to M and R. Our description of this characteristic corresponds to the analogous situation for the union:

sets are commutative under the intersection operation.

That is to say, under the intersection operation, the order of intersection is immaterial.

Now since the intersection of two sets is a set, we may consider the possibility of intersecting this set with yet another set. To illustrate, suppose $F = \{a, b, c, d, e, f\}$ and $S = \{b, d, f, h\}$. Then

$$F \cap S = \{a, b, c, d, e, f\} \cap \{b, d, f, h\}$$
$$= \{b, d, f\}$$

since b, d, and f are elements of both F and S. If $T = \{b, c, e, f, h\}$, then the intersection of $F \cap S$ with T would be

$$(F \cap S) \cap T = \{b, d, f\} \cap \{b, c, e, f, h\}$$
$$= \{b, f\}.$$

As before, the parentheses around F and S indicate the grouping of these sets to form the first intersection. Thus, intersection of sets may be formed successively one upon another just as unions may be formed successively. Parallel to our previous investigation of the union, we may pursue the question regarding the result of grouping these same three sets differently. The question then, might be: "How does $F \cap (S \cap T)$ compare with $(F \cap S) \cap T$?" To answer this, first observe that

$$S \cap T = \{b, d, f, h\} \cap \{b, c, e, f, h\}$$
$$= \{b, f, h\}.$$

Therefore,

$$F \cap (S \cap T) = \{a, b, c, d, e, f\} \cap \{b, f, h\}$$
$$= \{b, f\}.$$

This clearly gives the same result that we obtained above for $(F \cap S) \cap T$. In general, we have the associative property under intersection:

for sets A, B, and C, it is true that
$(A \cap B) \cap C = A \cap (B \cap C)$.

and we may simplify both of these expressions by dropping the parentheses:

$$(A \cap B) \cap C = A \cap (B \cap C) = A \cap B \cap C.$$

On reflection, this must be so. In all these cases, the final result is the set of all elements that all three sets have in common.

Set intersections will play an important role in our work with numbers and with geometric objects. In particular, when we discuss the rational numbers we will see that they figure very prominently, such as in reducing fractions and finding common denominators.

## Problems

14. If $A$, $B$, and $C$ are the sets specified below, illustrate the associative property under intersection by different groupings for $A \cap B \cap C$.

    a. $A = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$,
    $B = \{2, 4, 6, 8, 10, 12\}$, and $C = \{3, 6, 9, 12\}$.

    b. $A = \{2, 4, 6, 8, 10, 12\}$, $B = \{3, 6, 9, 12\}$,
    $C = \{2, 4, 7, 11\}$

    c. $A = \{2, 4, 7, 11\}$, $B = \{3, 6, 9, 12\}$ and
    $C = \{2, 4, 6, 8, 10, 12\}$.

15. If $A$ and $B$ are disjoint, what would be the intersection, $A \cap B \cap C$? What would be the intersection, $A \cap B \cap C \cap D \ldots \cap Z$?

## The Product Set

In the foregoing sections in this chapter, we have shown two different ways that a third set may be created from two given sets; one, by the union, and the other, by the intersection. Because a set is derived or produced from the given sets in each case, the "derived" set may be considered a product of the other two.

There is a third way of producing a set from two given sets. This is to form all possible pairs of the elements. The formation of such sets will be linked directly to multiplication of numbers as well as to graphing.

If $A = \{1, 2, 3\}$ and $B = \{a, b, c, d\}$, we pair 1 with each of the members of $B$ in turn; next, we pair 2 with each of the members of $B$, continuing thus until we have exhausted all possibilities. The set of all such pairs is called the product set or Cartesian product of $A$ and $B$. The abbreviation for the product set of $A$ and $B$ is $A \times B$.

Notice that in the pairs here, we allow an element to be paired more than once. To denote that 1 is paired with a, for example, we use the form $(1, a)$. Thus, all possible pairs in $A \times B$ are:

$$\begin{array}{llll}(1, a), & (1, b), & (1, c), & (1, d), \\ (2, a), & (2, b), & (2, c), & (2, d), \\ (3, a), & (3, b), & (3, c), & (3, d).\end{array}$$

Each pair in the product set, $A \times B$, consists of a pair of elements. The first element is a member of $A$ and the second is a member of $B$.

On the other hand, in the product set $B \times A$, the first element in each of its pairs is a member of $B$ and the second is a member of $A$. Thus, the possible pairs of $B \times A$ are $(a, 1)$, $(a, 2)$, ..., $(d, 3)$, and are distinct from those of $A \times B$. That is to say, for example, the pair, $(1, a)$ is not the same as $(a, 1)$,

$$(1, a) \text{ is a member of } A \times B;$$
$$(a, 1) \text{ is a member of } B \times A.$$

Therefore, an order is induced in these pairs by the way we manufacture them. Here is a case then, where an operation is not commutative. However, both $A \times B$ and $B \times A$ are Cartesian products. Germane to our immediate purpose is the number of pairs that we can evolve for either of the product sets. We shall expand on this idea in a later chapter.

An appropriate observation that we might make now concerns the kinds of objects we have produced as elements of the third set by each of the operations: the union, intersection, and Cartesian product.

The union is formed by including in the third set every member of the two given sets. For example, if

$$A = \{a, b, c\} \text{ and } B = \{d, e\},$$

then

$$A \cup B = \{a, b, c, d, e\}.$$

Each member of $A \cup B$ is one of the members of the "parent" sets.

The intersection also has this feature--each member of $A \cap B$ is one of the members of the "parent" sets. If

A = {★, ○, △} and B = {★, △, □, ⌓, ☽},

A ∩ B = {★, △}.

On the other hand, each member of a Cartesian product is a pair of elements, one selected from each of the "parent" sets. Thus, if

$$A = \{1, 2, 3\} \text{ and } B = \{\alpha, \beta\},$$

then $$A \times B = \{(1, \alpha), (1, \beta), (2, \alpha), (2, \beta), (3, \alpha), (3, \beta)\}.$$

The members of $A \times B$ are pairs, and are different kinds of animals, so to speak, than any one member of the "parent" sets.

## Broader Concept of a Union

In our discussion of the union, the concept of this operation was made on the basis of two disjoint sets. The reason for this restriction is that eventually we intend to link this concept to the addition of whole numbers. Actually, the definition of union does not have this restriction.

The union of A and B is the set
whose elements are members of A, or
members of B, or of both A and B.

That is to say, elements of $A \cup B$ are members of at least one of the two sets, A, B. With definition, the concept of a union is broadened to encompass joining sets that have members in common as well as sets that are disjoint. For example, if

A = {★, ○, △} and B = {★, △, □, ⌓, ☽},

then

$A \cup B$ = {★, ○, △, □, ⌓, ☽}.

Note that the common members ★ and △ are not listed more than once; this is in accord with our previous agreement on the specification of a set. The properties that we have noted before under the restricted operation still hold for the broader concept of union.

### Problems

16. For each pair of sets given below, find $A \cup B$, $A \cap B$, and $A \times B$.
    a. A = {a, b, c, d, e}, B = {c, e, f}
    b. A = {c, e, f}, B = {a, b, c, d, e}
    c. A = {a, b, c}, B = {a, b, c}
    d. A = {a, b, c}, B = {d, e}
17. If A is a set what is $A \cup A$?

## Summary of Properties

A summary of the properties for sets that we have mentioned in this chapter is catalogued below, where A, B, and C are sets. These are properties that will be particularly meaningful for us when we deal with numbers or sets of points.

1. Sets are COMMUTATIVE under union and intersection

$$A \cup B = B \cup A \quad \text{and} \quad A \cap B = B \cap A.$$

Note here that the order of operation is immaterial.

2. Sets are ASSOCIATIVE under union and intersection

$$(A \cup B) \cup C = A \cup (B \cup C) \quad \text{and} \quad (A \cap B) \cap C = A \cap (B \cap C).$$

Note here that the grouping for the operation is immaterial.

3. $A \cup (\ ) = A.$

Applications to Teaching

The concept of joining disjoint sets has been reported to be fairly easy for children to comprehend. Apparently, this is a word that is used occasionally in other situations. Sets of buttons, books, or other concrete objects may be joined with other sets of any concrete objects to communicate in a natural way the notion of a union.

The notion of a commutative operation can also be rendered in a concrete form such as books from the shelf joined with books on the desk and books on the desk joined with books on the shelf. In either case, the same set of books is in the union. The words "union" and "commutative" need not be introduced at this point.

The concept of intersection is not introduced until the second grade and formalized in the third grade. However, we can see that it has properties that are analogous to those for the union. We shall make use of the intersection in the next chapter on geometry as well as in our treatment of fractions.

The Cartesian product will be used here mainly in connection with multiplication and with graphing when we compose two numbers to locate points in the plane. The number line will be introduced here along with the presentation of whole numbers in Chapter 6. Eventually, the student will encounter the Cartesian product in terms of relations between two sets. The various correspondences between the elements state which elements of one set are related to which elements of another set. Then, a principal undertaking will be to study the characteristics associated with various kinds of relations. From this will evolve the important study of functions. Graphing, of course, gives a pictorial representation of relations. Thus, graphing will be a valuable support for establishing some of the underlying concepts of relations.

As we have hinted at in the text, the notion of a relative complement will be applied to our development of the concept of subtraction. In the presentation of these ideas, teachers have reported that they find it helpful to use the words "remove" or "left over" prior to the particular lessons.

## Exercises - Chapter 4

1. a. Find the union and the intersection of A and B if $A = \{1, 2, 3, 4, 5\}$ and $B = \{1, 3, 5\}$.
   b. If B is a subset of A, what is $A \cup B$?
   c. If B is a subset of A, what is $A \cap B$?
2. Explain how the union and the intersection of any set with the empty set agree with your findings in Exercise 1.
3. A recipe calls for separating egg whites from egg yolks and emphasizes the particular order these are to be added. Explain the implication of these directions for the cake-mixing operation.
4. State which of the following situations are associative.
   a. Putting peas and carrots together, and adding water.
   b. Eating hot dog, mustard, and coffee.
   c. Paying for groceries with a quarter, a dime, and a nickel.
   d. Putting kerosene with fire, and adding water.

*5. State why the intersection of two sets is always a subset of their union.

---

## Solutions for Problems

1. a., b., and c. are unique correspondences.
   d. is not unique; 1 corresponds to both -1 and 1; 4 corresponds to both -2 and 2.
2. The only one that does not define an operation is d.; 0 corresponds to both 0 and 1.

3. a. Multiplication by 3
   b. Squaring
   c. Multiplication by 0
   d. Getting the opposite of a number
   e. Adding 3
   f. Adding 3 and taking the opposite result
   g. Getting the reciprocal
   h. Classifying according to family

4. a. $\{1, 2\}$
   b. $\{1, 2, 3\}$
   c. $\{1, 2, 3\}$
   d. $\{1, 2, 3, a, b, c, d, e\}$
   e. $\{a, b, c, d, e, 1, 2, 3\} = \{1, 2, 3, a, b, c, d, e\}$; same as in d.
   f. $\{a, b, c, d, e, 1, 2, 3, \alpha, \beta, \gamma, \delta\}$
   g. $\{1, 2, 3, \alpha, \beta, \gamma, \delta\}$
   h. $\{a, b, c, d, e, 1, 2, 3, \alpha, \beta, \gamma, \delta\}$; same as in f.

5. a. 4c
   b. $A \cup \{\ \} = A$

6. $W \cup X = V$

7. a. Commutative property under union.
   b. Commutative property under union; only order of B and C is changed.
   c. Associative property under union.

8. a. Union of a set and its complement.
   b. Union of a set and its complement.
   c. Associative property under union; $E = B \cup C$ and $D = A \cup B$, so $A \cup E = D \cup C$ states that $A \cup (B \cup C) = (A \cup B) \cup C$.

9. The union of M and the empty set is M.

10. a. Normally commutative; this depends on the layout of the blocks, also on the location. If the location of the starting point is 3 blocks south of the north pole, then walking 3 blocks north, the north pole is reached. At that point, there is no westerly direction ; everywhere is south.
    b. Commutative.
    c. Not commutative.
    d. Not commutative.
    e. Not commutative.
    f. Commutative.

11. a. Intersection

b. Union

c. Intersection

d. Intersection

e. Intersection; in a later section of this chapter, "Broader Concept of a Union", it will turn out that this is also the union of $A$ and $B$.

f. Intersection

g. Union

h. Intersection

i. Union

12. a. $\{h, o, w\}$

b. {blue-eyed, blond}

13. If $B$ is a subset of $A$, then $A \cap B = B$

14. a. $(A \cap B) \cap C = \{2, 4, 6, 8, 10, 12\} \cap \{3, 6, 9, 12\} = \{6, 12\}$
$A \cap (B \cap C) = \{1, 2, 3, \ldots, 12\} \cap \{6, 12\} = \{6, 12\}$.

b. $(A \cap B) \cap C = \{6, 12\} \cap \{2, 4, 7, 11\} = \{\ \}$.
$A \cap (B \cap C) = \{2, 4, 6, 8, 10, 12\} \cap \{\ \} = \{\ \}$.

c. $(A \cap B) \cap C = \{\ \} \cap \{2, 4, 6, 8, 10, 12\} = \{\ \}$
$A \cap (B \cap C) = \{2, 4, 7, 11\} \cap \{6, 12\} = \{\ \}$.

15. The empty set in either case.

16. a. $A \cup B = \{a, b, c, d, e, f\}$; $A \cap B = \{c, e\}$;
$A \times B = \{(a, c), (a, e), (a, f), (b, c), (b, e), (b, f), (c, c), (c, e), (c, f), (d, c), (d, e), (d, f), (e, c), (e, e), (e, f)\}$; the pairs may be listed in a different order; for example, $(d, e)$ may be listed as the first pair. However, the order of the elements within each pair must be observed. $(d, e)$ is a member of $A \times B$, but $(e, d)$ is not a member of $A \times B$.

b. $A \cup B = \{a, b, c, d, e, f\}$; $A \cap B = \{c, e\}$;
$A \times B = \{(c, a), (c, b), (c, c), (c, d), (c, e), (e, a), (e, b), (e, c), (e, d), (e, e), (f, a), (f, b), (f, c), (f, d), (f, e)\}$;

c. $A \cup B = \{a, b, c\}$; $A \cap B = \{a, b, c\}$;
$A \times B = \{(a, a), (a, b), (a, c), (b, a), (b, b), (b, c), (c, a), (c, b), (c, c)\}$;

d. $A \cup B = \{a, b, c, d, e\}$; $A \cap B = \{\ \}$;
$A \times B = \{(a, d), (a, e), (b, d), (b, e), (c, d), (c, e)\}$

17. $A \cup A = A$

Chapter 5

# ELEMENTS OF GEOMETRY

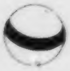
## Introduction

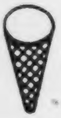
The last chapter dealt with operations on sets to form new sets. The ground now has been prepared for considering sets of points using the set operations to generate various kinds of geometric figures. Besides being a natural direction in which to move at this juncture, geometric figures and certain of their properties will be useful in extending number ideas in the chapters ahead.

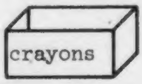
We shall begin our approach to geometry in a way that will be most helpful to teachers of small children. That is, we will consider concrete objects and abstract from them certain desired geometric information. We shall then shift to a more mathematically logical approach

## Geometric Solids

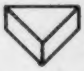
Observation of concrete objects such as those shown below will be helpful for an understanding of abstract representations of figures.


ball

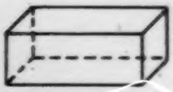

ice cream cone

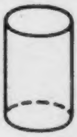


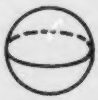
box.


block

Discussion of the characteristics of these shapes facilitates familiarity with some of the vocabulary associated with them.


(a)


(b)

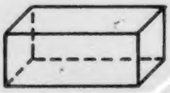

(c)

The above drawings are examples of typical representations of geometric solids. There may be some difficulty in visualizing the 3-dimensional nature of the figures since the drawings are restricted to two dimensions. The dotted lines are included to aid perception. They represent parts of the figures which would not be visible from this vantage point.

In all of these figures, the "inside" is not filled. The object identified by (a) looks like a block. It is not like a block in terms of being composed of matter such as wood. It is shaped like a block but is hollow. Physical objects which can be associated with the geometric solids illustrated are a shoe box, (including the lid), an empty oatmeal box with the lid on, and a balloon. Thus, the word "solid" in "geometric solid" has the mathematical meaning of "three-dimensional" rather than the common usage of "firm" or "not hollow".

The figures (a), (b), and (c) above can be abstracted from numerous physical objects which are available to the teacher and children. Each has characteristics which convey the ideas we want to teach. For example, by looking at and touching models of (a), children will learn to recognize "straight" and "flat" objects with "corners". In (b), they will feel a "rounded" object which also has "edges" and "flat" parts, but no "corners". The third figure illustrates a "rounded" surface without edges or corners.

For our purposes in developing some basic concepts and vocabulary, we will concentrate only on Figure (a). The subject of geometric solids will be developed more thoroughly in a later chapter.

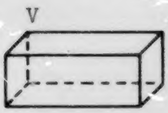


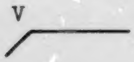
(a)

This "box" (more formally, a rectangular prism) is made of six flat surfaces which are called faces of the prism. The face of a geometric solid is a flat surface of the solid.

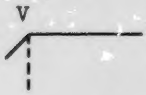
Where two faces meet is an edge of the solid. Each face of this figure has a boundary of four edges. The "skeleton" of the prism is made up of twelve edges.

One other characteristic which we wish to identify in the above solid is that it has "corners" where three edges come together. Each is a vertex (plural: vertices) of the prism. Note that any two of these three edges would meet at the same place and form the same geometric figure. Thus the two figures to the right below



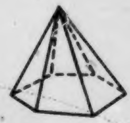


and

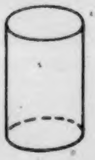


equally well locate the vertex of the prism identified by V. Thus, a vertex may be determined by the meeting of two edges of a face. A point of a geometric figure may sometimes be designated a vertex, however, even though it is not the meeting of two edges of a face. This is the case, for example, with the vertex of a cone.

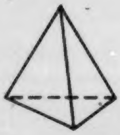
Problems*



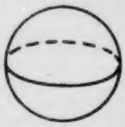
1. How many faces does this solid contain?
2. Which of the figures have edges but no vertices?



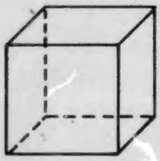
(a)



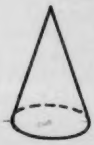
(b)



(c)



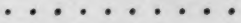
(d)

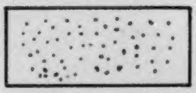


(e)

Points and Paths

The basic ingredient of all geometric configurations is what is called a point. We rely on intuition to arrive at the concept that an edge of a solid is made up of an aggregate of points; so is a face.





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* Solutions for problems in this chapter are on page 136.

A vertex, where two or more edges come together, is a point. The tip of a pencil or the period at the end of this sentence helps us visualize what a point is. We might consider that the dot of an "i" is a better picture of a point than the period, because it is smaller. All of these are visual aids to assist us in conceptualizing the nature of a point.

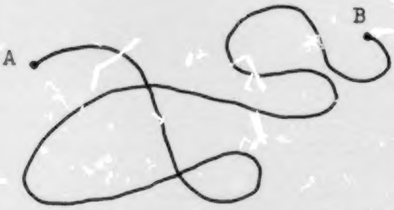
The difficulty is that a point is an idea rather than a physical object. We try to develop the idea through pictures of ever-shrinking dots as we have done on the preceding page. Through this process, we arrive at the description of a point as an exact location.

This is not, however, a definition of point in the formal sense. If we say a point is an exact location, "exact location" must be understood. The dictionary might define location as a "position in space". Position in space might refer us back to point. If none of these words were meaningful to us, the dictionary would hardly clarify matters. However, the circularity in dictionary definitions is necessary because there is only a finite number of words accessible in the dictionary. Eventually some word in the chain of definitions must reappear. Implicit in this is that at least one word in the chain must be simply understood so that others may be defined in terms of it. "Point" is such a word in geometry. In a sense it is the "first" word in the vocabulary of geometry, and we say it is an undefined term.

Once the concept of point is understood, we will again rely on representing points by marks on paper to facilitate discussing them. They are commonly labeled by capital letters. The drawing represents point P, or simply P, by which a point is understood.

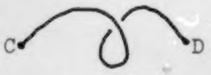
.P

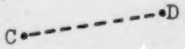
Every geometric figure is a set of points. A curve is a set of points followed in moving from one point to another.



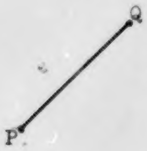
Thus the "wiggly" drawing represents a path from A to B. It is evident that there are other curves from A to B; indeed, there are infinitely many.

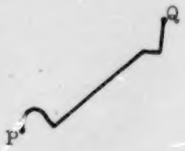
Inherent in the notion of path is the idea of continuity. There may not be gaps in a path. Neither of the drawings below is a path from C to D.

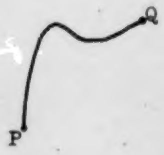




According to the strict mathematical definition, curves do not have to be continuous; however, we will consider only those that are. Hereafter, by "curve" we shall mean a continuous curve. Portions of the path or the entire path may be straight. As a path may be used to specify the set of points in a curve, any of the following figures represents a curve from P to Q.

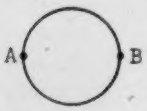


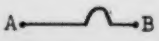


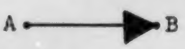


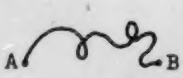
Problem

3. For each of the following figures state whether it represents a curve from A to B.



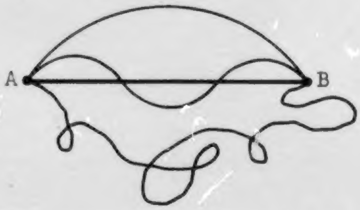






Line Segment

Many geometric figures are built out of straight curves. Examples and prior experiences are relied on to convey the meaning of straight.



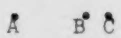
We have pointed out that the possible paths from A to B are unlimited or infinite. Each is a set of points. The one straight path that can be considered from A to B describes the line segment whose end points are A and B. It is denoted by $\overline{AB}$ or $\overline{BA}$ ($\overline{AB}$ is read: "segment AB"). The notation, $\overline{AB}$, denotes the segment with A and B as endpoints; the notation, $\overline{BA}$, denotes the segment with B and A as endpoints. Since both $\overline{AB}$ and $\overline{BA}$ denote the same segment, the order in which the endpoints are names is irrelevant. This is consistent with the properties of sets since a set is the same no matter how its elements are ordered. With the concept of line segment, we can now identify an edge of a rectangular prism as a line segment.

Problems

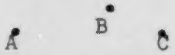
4. Represent $\overline{AB}$ such that it can also be named as the union of $\overline{AQ}$, $\overline{QM}$ and $\overline{MB}$.

5. In the same $\overline{AB}$ of problem 4, assume Q is between A and M. State all other possible relationships of one point being between two other points.

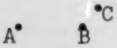
By $\overline{AB}$ is implied the set of points, A, B, and all points between them. Thus the notion of betweenness is intuitively derived. However, there may be need to clarify what is meant by "between." If A, B, C, are three points as indicated in (a), it may be quite natural to consider that B is between A and C. Even if the points were as



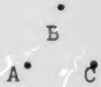
(a)



(b)



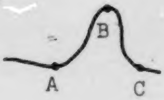
(c)



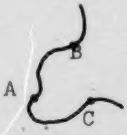
(d)

(b) or (c) one might concede that B is between A and C. But if the three points were as in (d), the question as to which point is between which other two points is not so easily resolved.

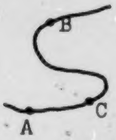
Implicit in the decision as to which point is between two others apparently is a curve connecting these points. If a curve passes through the three points as in (e), then B is between A and C.



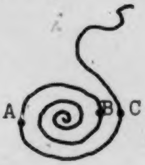
(e)



(f)



(g)



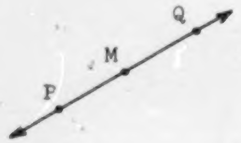
(h)

In (f), A is between B and C, and in (g), C is between A and B. This, of course, can be done for points which we may have considered sufficiently clear with regards to betweenness. Thus, in (h), A is between B and C. Eventually, a number will be associated with each pair of points. We will call this number the distance from one point to the other. Betweenness can then be stated in terms of distances. Even with this definition, a curve is involved in the concept of distance. The common sense interpretation of betweenness, when no curve is specified, is simply that the points are to lie along a straight path. When we say that a point is between two others, it will be our understanding then, that the three points are all on the same line.

## Line and Ray

Once a line segment is defined by the location of its two endpoints and all the points between them, the straightness of it determines two directions. If we imagine extending a given segment infinitely far in both of these directions, we conceive of a geometric *line*.

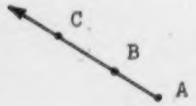
A circularity will be noted in defining a line and betweenness. A *line* is conceived of as an extension of a segment, and a *segment* is defined as the set consisting of two endpoints and all the points between. On the other hand, *between* is stated in terms of points on a line. This circularity is unavoidable in definitions, and ultimately, we must accept these notions as primitive and undefined. Thus, in geometry, a line is accepted simply as: "a certain set of points".



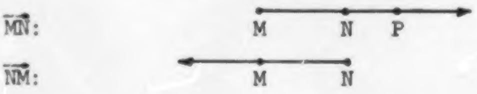
The drawing represents the line formed by extending $\overline{PQ}$ in both of its determined directions. The arrowheads are used to indicate that the extension is infinite. We adopt the notation $\overleftrightarrow{PQ}$ for the line containing the two points P and Q in order to distinguish it from segment PQ written as $\overline{PQ}$. We could also refer to the line as $\overleftrightarrow{PM}$, $\overleftrightarrow{MQ}$, $\overleftrightarrow{QM}$, and so on. In general, any "two points in the set of points" in the line may be used to name it. Again, order does not matter.

It is important not to use this terminology loosely. A line has no endpoints, while a line segment must have two endpoints.

A third possible straight configuration can be formed by extending a line segment in one direction only, as indicated below.



The figure so obtained is a ray. The one shown can be formed by extending $\overline{AB}$ through B or $\overline{AC}$ through C. This ray cannot be formed by an extension of $\overline{BC}$ in one direction, however. The simplified notation of this ray is $\overrightarrow{AB}$ or $\overrightarrow{AC}$. In contrast with the notations for segments and lines, in naming rays the order of points is significant. A ray has one endpoint, and it is named first. The second letter can name any other point in the ray. As is indicated below, $\overrightarrow{MN}$ and $\overrightarrow{NM}$ are not equal sets. There are



common points in the two sets: any point in $\overline{MN}$ is a member of both rays. However, P is in $\overrightarrow{MN}$ but is not a point of $\overrightarrow{NM}$.

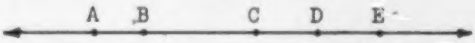
Note that the arrow in the nomenclature $\overrightarrow{MN}$ designates which is the endpoint of the ray; it is not the intention to convey the orientation of the ray as it appears. In fact, it would be impossible to orient the arrows in conformity with all possible orientations of rays.

Problems

6. Represent $\overrightarrow{PR}$ and show Q between P and R. Which of the following denote the same ray?
   $\overrightarrow{PQ}$, $\overrightarrow{QP}$, $\overrightarrow{QR}$, $\overrightarrow{PR}$, $\overrightarrow{RP}$, $\overrightarrow{RQ}$

7. a. What is the implication of the statement $\overrightarrow{AB} = \overrightarrow{CB}$ ?
   b. Does $\overrightarrow{BA} = \overrightarrow{BC}$ have a similar connotation?

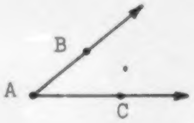
8. Referring to the drawing below, rename the sets in simple notation.



   a. Union of $\overline{BC}$, $\overline{CD}$ and $\overline{DE}$
   b. Intersection of $\overrightarrow{AB}$ and $\overrightarrow{BC}$.
   c. Intersection of $\overrightarrow{CA}$ and $\overrightarrow{ED}$.
   d. Intersection of $\overrightarrow{CD}$ and $\overrightarrow{DC}$.
   e. Union of $\overrightarrow{EA}$ and $\overrightarrow{BC}$.

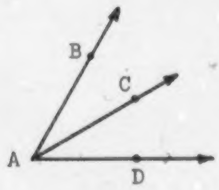
## Angle

Another fundamental geometric figure recognized in many familiar shapes is an angle. The formal definition is: an angle is the union of two rays which have a common endpoint but which are not subsets of the same line.



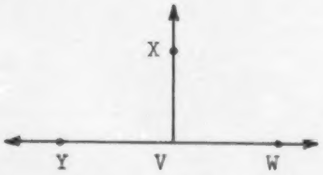
The example shown is the union of $\overrightarrow{AB}$ and $\overrightarrow{AC}$. Their common endpoint is said to be the vertex of the angle. Recall that vertex also applies to geometric solids and their faces. In each case it is the intersection of appropriate edges. Similarly, here, the vertex of an angle is the intersection of the two sets of points in the rays. The rays are called the sides of the angle.

Our angle is denoted by $\angle BAC$ or $\angle CAB$, where the middle letter identifies the vertex. The other two letters name one point distinct from the vertex on each of the two sides. Often, simply $\angle A$ will be written instead of $\angle BAC$. This notation cannot be used if more than one angle is drawn at vertex A.



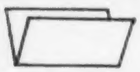
It would not be clear by $\angle A$ which of $\angle BAC$, $\angle CAD$, or $\angle BAD$ were meant in the figure above.

A special angle that makes frequent appearance in mathematics is a right angle. No formal definition is given at this time. Instead, we will describe what is meant by a right angle in much the same way that you will convey the concept to your students.



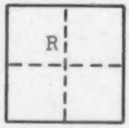
The above drawing represents two right angles, $\angle YVX$ and $\angle WVX$. This is one means of describing them. Two right angles "fit together to form a line". Also, in our example, if this page were folded along $\overline{VX}$, then $\overline{VY}$ and $\overline{VW}$ would coincide.

If a piece of paper were folded twice, as the drawing indicates, and it were then unfolded, the creases suggest segments of two lines whose intersection is the point R. Thus, R is the vertex of four right angles whose sides are the extensions of appropriate pairs of creases.



Fold 1

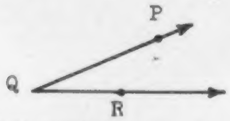
Fold 2



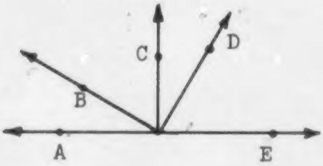
Unfolded with
creases dotted

Problems

9. a. Name the angle shown in three ways.
   b. Identify the sides of this angle.



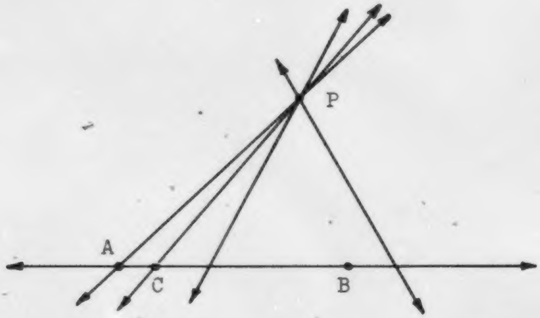
10. Identify all angles which appear to be right angles



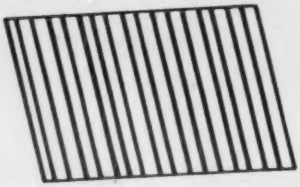
11. Can $\overrightarrow{VX}$ and $\overrightarrow{WX}$ be sides of an angle?

## Plane

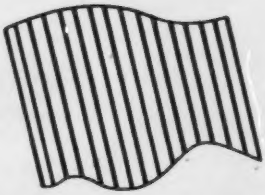
We have considered all the fundamentally distinct figures that can be elicited from a line segment if we retain the characteristics of straightness and one-dimensionality. Now let us take a line, $\overleftrightarrow{AB}$, and a point P, not in the set of its points. It has been implicit



in talking about a line that it is an infinite set. It was mentioned that a path is not allowed to have holes in it, and this is also true of a line. The conception of a plane we now want is that of a set of points. This set will be the union of all possible lines determined by P and each of the points $\overleftrightarrow{AB}$, only four of which have been shown above. It may be difficult to visualize that this will generate an entire plane without "slits". If there is a "slit", between $\overleftrightarrow{PA}$ and $\overleftrightarrow{PC}$, for example, then all that needs be done is to think of connecting P to more points between A and C until the space is filled. The reason we have chosen to generate a plane with a line and a point is that it assures a plane as a result. It is also consistent with the fact that a line segment was generated by a point, another point, and all points between them. If we had, instead, considered "sweeping" a line infinitely far in directions other than the two it determines, how can we be sure that it is flat, not wavy?



Sweeping out a plane



A wavy sweep

We would have to use words like "flat" or "without rising or dipping." But these descriptions imply the very concept we are defining, and we would be caught in a circular definition. As before, a plane may be accepted as one of the primitive undefined elements, and this is done in geometry. Simply, "a plane is a certain set of points".

We have shown how a plane may be generated by a line together with a point that is not in the line. Thus we say, a line and a point not in the line determine a plane. That is, there is a plane of which they are both subsets, and there is no other. As a consequence, we can also say that two lines which intersect in a proper subset of each line also determine a plane. The reason is that if two lines intersect in a proper subset of each, the intersection consists of one and only one point. Then, on any one line there is a point that is not on the other line. Using this point and the other line, the plane is determined. Notice further that if two points in a line lie in a given plane, then the entire line is a subset of the plane.

In teaching children, the definition of a plane either as a primitive element or as the result of generating lines would obviously be too esoteric. To express the meaning of plane, such objects as the floor, tabletops, and faces of a block are examined and felt. The notion of the infinite extent of the plane is approached by thinking in terms of an ever-expanding tabletop and so on.

## Problems

12. Describe the geometric figures that would be generated by:
    a. all possible line segments determined by three points A, B and C.
    b. all possible line segments determined by the set of points in a line and a point not in the line.

13. Why is it impossible for the intersection of a line and a plane to contain exactly two points?

## Space

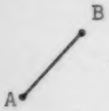
Space is simply the set of all points. The set of all points in a plane is a space of two dimensions and a line is a one-dimensional space. In other words the term "space" can be used within different frames of reference.

The usual connotation of space is the set of all points in a three-dimensional extent. We could generate three-dimensional space (3-space) in a manner similar to our definitions of line and plane. We could say that "space" is the set of points obtained by considering a plane, a point not in a plane, and all possible lines determined by the point and each point in the plane. In the context of 3-space, a plane is a subset of space; so are a line, a point, and all figures made up of them.

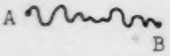
The notion of space in the more general sense, as simply a set of all points, is extended to the branches of mathematics other than geometry. Thus, in probability, the set of all possible outcomes of a certain situation is described as the sample space. The meaning of space is generally determined by the context in which it is used. Unless otherwise indicated, space in this text will refer to infinite, three-dimensional space.

## Simple Closed Curves

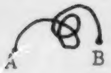
In our discussion of segments, we considered paths between two points and observed that each of the paths describes a curve. A path thus specifies a set of points known as a curve from A to B. When A and B coincide, the curve is said to be closed. Thus, each of the diagrams illustrated represents a curve. The ones appearing on the second row are closed curves.



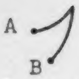
(a)



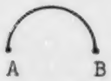
(b)



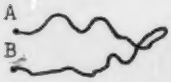
(c)



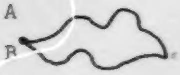
(d)



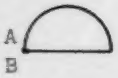
(e)



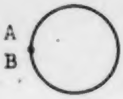
(f)



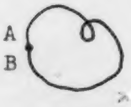
(g)



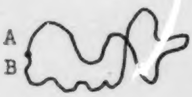
(h)



(i)

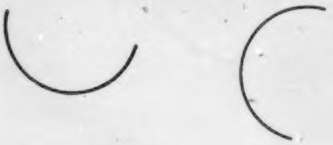


(j)

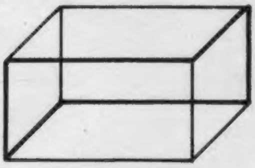


(k)

A semi-circle as in (e) is not a closed curve, and this accords with our intuition. As a curve is a set of points, it is possible that the union of two such sets might define a curve; moreover, it may be that the curve so defined is closed. This is the case for example, in the union of a semi-circle and a particular diameter as in (h), or in the union of a semi-circle and another semi-circle to form the entire circle, as in (i). On the other hand, the union may be simply unconnected pieces as in



Of the closed curves that we have drawn, the first three are distinguished from the last two. None of the first three curves crosses itself. To describe the fact that the curve does not cross itself, we say it is simple.

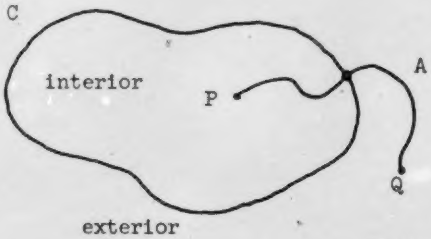


The curve that follows the heavily drawn edges of the box illustrated here is a simple closed curve. It is simple because it does not cross itself, and it is closed because a point that is considered an initial point of this curve may also be its terminal point. That is, we can start and finish a complete "journey" at the same point. However, the simple closed curves that we shall be concerned with chiefly are those that are contained in a plane. Hereafter, by "simple closed curves" we shall mean those in a plane, unless we specify otherwise.

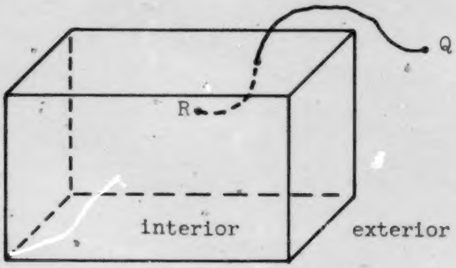
Simple closed curves have the important property of separating the rest of the plane into two disjoint subsets, the interior (the subset of the plane enclosed by the curve) and the exterior. Thus, with a simple closed curve there is a natural partition of a plane into three disjoint subsets:

(1) the set of points that are enclosed by the curve
(2) the set of points that are on the curve
(3) the set of points that are neither enclosed by the curve nor on the curve.

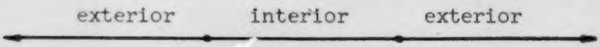
With the separation, any curve in the plane connecting a point of the interior with a point of the exterior necessarily intersects the simple closed curve. This is illustrated by the figure below, where C is the simple closed curve, P is an interior point, Q is an exterior point, and A is a plane curve connecting P and Q.



Analogous to this separation property of a simple closed curve in the plane is the separation property of a simple closed surface in three-dimensional space (3-space). The boundary of a rectangular box (formed by the faces) is an example of a simple closed surface. The surface separates the rest of 3-space into two disjoint subsets, the interior and the exterior, so that any curve in space connecting a point of the interior with a point of the exterior necessarily intersects the simple closed surface. This is shown in the figure on the next page.



The situation in 1-dimensional space (for example, on a line) is not quite analogous to either of these, but we can still define a set of points forming a boundary that would specify interior and exterior points. The set of points that lie between the two points is the interior, and the points on either side constitute the exterior.

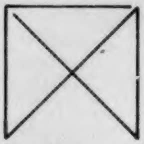


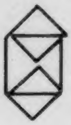
We said that the situation in the 1-dimensional case is not quite analogous to cases in higher dimensions. However, it is still true that a point in the interior cannot be connected with a point in the exterior without intersecting a boundary point. It is also true that it is possible to connect an interior point with an interior point without intersecting a boundary point. In the higher dimensions it is possible to connect two exterior points without passing through the boundary; this is not always the case for one-dimensional space. For example, if A and B are the boundary points, and if P and Q are in the exterior on different sides of the boundary, then we must pass through the boundary to connect them.

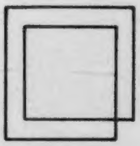
Even so, there are enough similarities to consider these situations analogous.

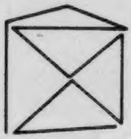
Problems

14. Which of the following curves are simple?


(a)


(b)


(c)


(d)

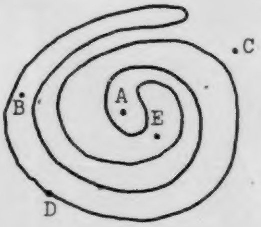
15. Which of the above curves are closed? Which is simple and closed?

16. Which of the curves in Problem 14 is a union of two simple closed curves?

17. If the letters of the alphabet were printed in block type without serifs (no "tails"), which letters indicate simple closed curves? Which are simple but not closed?

A B C D ···

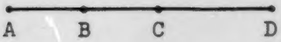
18. a. Can a curve be drawn from A to B without crossing the given curve? from A to C? from A to D? from A to E? State the reason for each case.



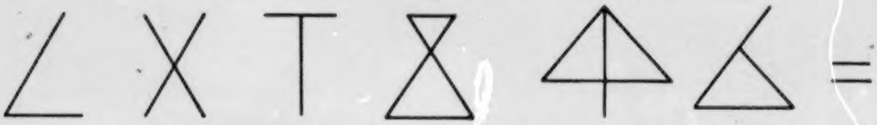
b. From which point in the above diagram is it impossible to draw a curve to any other point without intersecting the curve?

Polygons

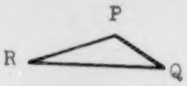
An important class of simple closed curves is the class of polygons. A polygon is a simple closed curve that is a union of line segments. Not all unions of segments form simple closed curves. For example, the union of two segments may again be a segment. In the picture below, the union



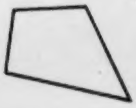
of $\overline{AC}$ and $\overline{BD}$ is $\overline{AD}$; the union is simple, but not closed. Nor is any of the figures below a simple closed curve; each is a union of line



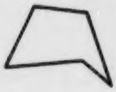
segments. Triangles, quadrilaterals, pentagons, and so on, are examples of polygons. Note that segment AD above contains many other segments. For example, $\overline{AB}$ is contained in $\overline{AD}$, $\overline{BC}$ is contained in $\overline{AD}$, $\overline{AD}$ is contained in itself, and so on. Likewise, with segments of a polygon, segments are contained in segments. If a segment of a polygon is contained in no segment other than itself, then this segment is called a side of the polygon. For example, $\overline{PR}$ is a side of the triangle shown below.



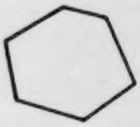
triangle



quadrilateral



pentagon



hexagon

A polygon of three sides is a triangle; four sides, a quadrilateral; five-sides, a pentagon; six sides, a hexagon; and so on. The endpoints of the sides are the vertices of the polygon. Note that each vertex is a common endpoint of two sides. Note also that the number of sides is the same as the number of vertices.

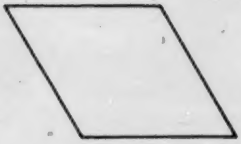
Rectangles are special kinds of quadrilaterals. Each corner of a rectangle determines a right angle. Intuitively, each corner may be "matched and fitted" onto each of the others. In turn, squares are special kinds of rectangles. Intuitively, each side of the square may be "matched and fitted" onto each of the others. Thus, in the family of quadrilaterals, subfamilies are identified. The rectangles

rectangle

square

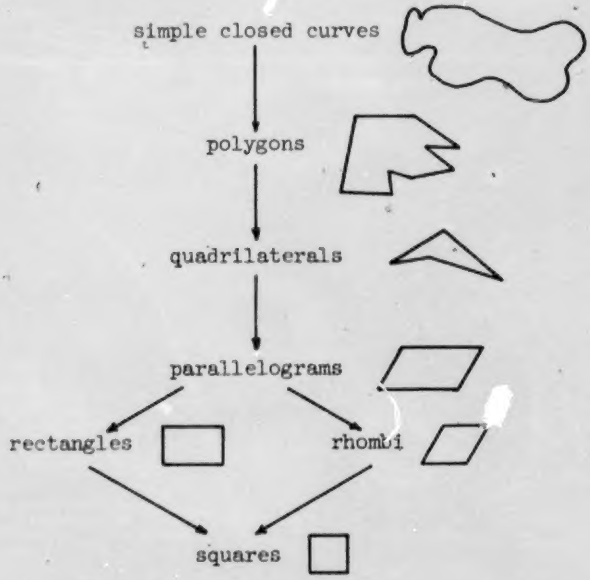
constitute a subfamily of the quadrilaterals and the squares constitute a subfamily of the rectangles. Another subfamily of the quadrilaterals are the parallelograms. Their opposite sides are segments of lines which are on the same plane and which do not intersect. As rectangles also possess this characteristic, rectangles are a subfamily of parallelograms. Another subfamily of the parallelograms are the rhombi (singular: rhombus). Each side of a rhombus may be "matched and fitted" onto each other side. So a square is both a special kind of a rectangle and a special kind of a rhombus.

parallelogram



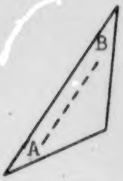
rhombus

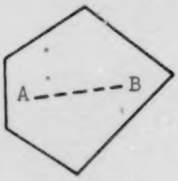
By this kind of classification, we get a generic chain that may be indicated by the following diagram.

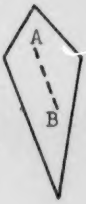

simple closed curves
polygons
quadrilaterals
parallelograms
rectangles
rhombi
squares

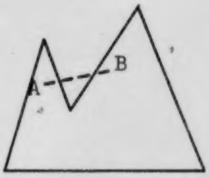
Some classification is made of other kinds of polygons although it is not as extensive as with quadrilaterals. Thus, for example, triangles may be specialized according to a variety of criteria as right triangles, isosceles triangles, equilateral triangles, etc.; and special polygons may have characteristics qualifying them as regular polygons.

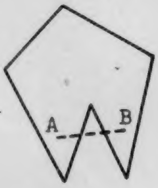
Convex polygons are of increasing interest nowadays because of their applications. A polygon is said to be convex if the segment determined by any two interior points lies entirely in the interior. That is to say, there is no point of the segment that is not an interior point of the polygon. Thus, polygons such as the three at the left below are distinguished from the others at the right. No two sides of a convex polygon are "dented" inwards. This notion of convexity applies equally to other simple closed curves that may be considered to be convex. If we pick any two points of a given set to be endpoints of a segment and find that every point of the segment is contained in the given set, then the set is said to be convex. Thus, the interior of a convex polygon is a convex set of points. The union of the convex polygon with its interior is also a convex set of points. Notice, however, that the polygon itself is not a convex set. The definition of convex polygon stated above identifies a polygon as convex if its interior is a convex set. The interior of a rectangular box and the interior of a ball are examples of convex sets of points. By analogy with the concept of convex curves, the rectangular box and the ball specify convex surfaces.











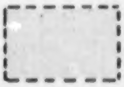
## Regions

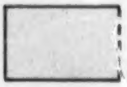
Since a polygon is a simple closed curve, it is the set of points on the curve. These points should be distinguished from the set of points enclosed by the curve which we call the interior; the two sets are disjoint. A circle is also a simple closed curve, and it also has an interior. The union of a simple closed curve and its interior is called a region. We refer to a triangular region, rectangular region, polygonal region, or circular region, etc., indicating that the simple closed curve is a triangle, rectangle, polygon, circle, etc. The union of a simple closed surface and its interior is also a region.

To denote a plane region in a diagram, the interior of the simple closed curve is usually shaded. To denote the interior only, the interior is shaded, but the polygon is drawn in dashed outline as is shown in the figures below.

rectangle

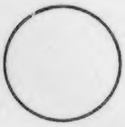
rectangular region

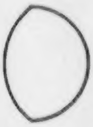

interior of
rectangle

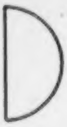

union of interior
of rectangle and
part of rectangle

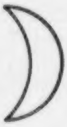
## Problems

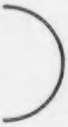
19. Which of the following represent convex simple closed curves? For each that is not, state how it fails to be one.

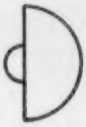

(a)


(b)

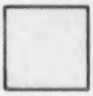

(c)

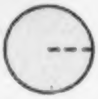

(d)


(e)


(f)

20. Which of the following represent convex sets? If not, why not?


(a)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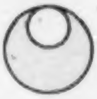
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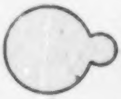
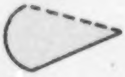

(d)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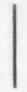
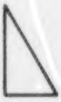

(f)

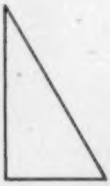
21. Which polygon is always convex as soon as the number of sides is specified?

Congruent Segments and Regions

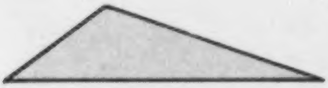
When we talked about squares, rectangles, and rhombi, we referred to a "matching and fitting" process without explaining precisely what was meant. We have relied on the intuitive concept of congruence as "having the same size and shape", and we shall continue to rely on intuitive concepts until we develop the notion of measures in a later chapter.

Geometric figures may have the same shape without being the same size, or they may have the same size without having the same shape. It is clearer to see what we mean by the former than by the latter. For example, the triangles below have the same shape but not the same size.





If two figures are in the plane, we may have some difficulty deciding what is meant by the figures having the same size but not the same shape. The question is whether we mean by "size" the total area occupied by the figure or the length of the boundary. The question is more troublesome with higher dimensional figures. Usually in the plane, the area is meant. Thus, the figures below have the same size although not the same shape.



With segments, the task is easier; one segment is always the same shape as another. To decide whether two segments are of the same size, we can make a tracing of one and try to match it to the other. If they match, we consider the segments to be congruent. It is, in this sense, of course, that markings on a ruler perform the function of movable copies of segments.

When we discuss measurements later we shall investigate more closely the role of the ruler and the notions of length, area, etc. At this time, we will extend the notion of a movable copy in helping to decide whether or not two regions are congruent. We can make a tracing of the boundary of one region and try to match this tracing to the other region. If the tracing matches, then the boundaries are congruent and the regions are congruent. The movable copy is needed because the geometric figures to be compared are sets of points, and as such, have fixed locations. Clearly, we cannot continue this matching process too long. A copy of a solid may not be matched and fitted into another solid. A more refined concept of congruence is not attempted until the children study geometry from a more formal standpoint.

In the light of congruence, we may restate the requirements of special geometric figures. For example, any two edges of a cube are congruent segments, and any two faces of a cube are congruent regions. Similarly, we can note the congruent sides of a parallelogram and so on.

Problems

22. Restate the requirements of a square in terms of congruence; of a rhombus.

23. What is the name of a triangle any two of whose sides are congruent?

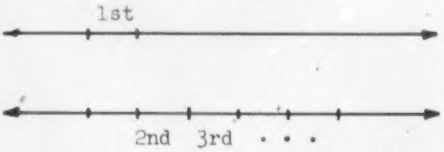
24. What are the congruent regions of a cylinder?

25. If a point of a segment determines two congruent segments so that the union is the given segment, what is the name of this point? What is the intersection of the two segments?

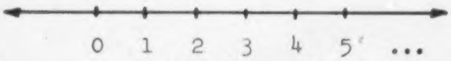
26. Divide an equilateral triangular region into two congruent regions.

## The Number Line

Congruent segments give us a way of relating numbers with points on a line. This is the case with the number line. Given any two points on a line a segment is determined. We can continue to mark off points, one after another so that each segment is congruent to the first.



The points may be labelled 0, 1, 2, 3, 4, ..., in the order of the whole numbers. Although one can assign these labels from right to left, conventionally we proceed from left to right. When points are labelled thus, the names of the points are called the coordinates of the points, and the line together with its coordinates is called the number line.



The Number Line

The number line thus gives us a 1-1 correspondence between the set of endpoints of congruent segments and the set of whole numbers. That is, each endpoint is associated with one and only one whole number, and each whole number is associated with one and only one endpoint of the congruent segments on the line. This device is quite useful for us. It enables us to visualize the order of numbers by the position of corresponding points on the line. We will later connect operations in arithmetic with operations on the number line.

### Problems

27. What is the smallest whole number represented on the number line?
28. What can you say about every number represented by a point on the number line that lies to the right of a given point?

Applications to Teaching

Logically, as geometric figures are made up of points, one should begin the study of geometry with the concept of what constitutes a point. Lines, curves, regions, planes, solids, and spaces may be generated from a point.

Thus, from two points and all the points between these two evolves a line segment having the two points as endpoints of the segment. The extension in both directions past the endpoints defines a line. By all possible "connections" of a point that is not on a line with all points of the line, a plane may be generated. (By "connection", we mean here the entire line that passes through the two points.) By all possible "connections" of a point that is not on a plane with all points of the plane, three dimensional space may be generated. All of these are sets of objects which we call points: each set of points has subsets whose elements are also points. Thus, a segment is a subset of a line, a line is a subset of a plane, and so on.

There are certainly many subsets of sets of points that we have not mentioned. For example, curves which are subsets of a plane are plane curves (such as circles, rectangles, or some wiggly plane curves). If a curve is not contained within any one plane then it is a space curve (such as the curve traced by a bed spring). Later, points in a curve will be associated with points in a line or in a segment on a one-to-one basis; that is, a 1-1 correspondence will be established between a point on a curve and a point on a line (or segment). Then, a curve will be viewed in terms of an operation or function of points on a line.

In summary, geometric configurations are sets of points or unions of such sets. A point is a set with a single member. A segment is the union of single member sets. The union of two points is a set and thus, the two points constitute a geometric configuration; so do a point and a curve; and so on. From the union of certain segments or curves we obtain such familiar figures as triangles, rectangles, circles, pyramids, cones, prisms, and spheres.

Despite the logical basis, the sets of geometric objects that children have to manipulate are sets of three-dimensional objects.

These are the concrete objects which provide children with experiences from which they can abstract the mathematical concepts. For this reason, we begin with models of solids. From the models, we identify faces, edges, and vertices. Once identified, we can use these primitive elements to construct other geometric figures. For example, "skeletons" of pyramids and prisms are unions of certain line segments.

Eventually, other kinds of primitive elements will be introduced to serve as building blocks for various geometric figures. These building blocks will be objects called simplexes. They include figures such as segments, triangular regions, and triangular pyramids. The configurations formed by the union of such elements are the complexes such as triangles and "skeletons" of pyramids. For the study of the complexes, what can be learned about simplexes will be extremely helpful although not all problems about complexes can be answered by relating complexes to the building blocks. Moreover, certain kinds of complexes give rise to special sets of points called convex sets which play a significant role in the branch of mathematics called linear programming. Linear programming has many applications in business and in the physical and social sciences. The contact which the children at this level have with simplexes and complexes are mainly in terms of polygons and other simple closed curves or solids consisting of edges. Such experiences will form a basis for future experiences in mathematics.

Of particular interest are complexes that are closed figures. Such complexes may specify where solutions to certain existing problems may be found. Closed figures do not have points that can be designated as the initial point and the endpoint. A circle, an oval, a triangle, a figure-eight, the surface of a rectangular box are all examples of closed figures. For children, the approach to closed figures is entirely geometric. It must be emphasized that any closed figure that does not lie in a plane is called a "solid", even though it is hollow. For example, a rectangular "box" consisting only of the faces is a "solid"; the "skeleton" of a rectangular box is a "solid".

It is a good idea to display a set of wooden models that are not too small and encourage the children to examine and handle them for several days before beginning the chapter, "Recognizing Geometric Figures." Tracings of the faces of the solids may be made on a large sheet of paper and displayed so that the children may match a face of a solid to its tracing. Wire or stick models of polygons which are congruent to boundaries of faces may be used for the same purpose. Construction of solid figures may emphasize that many solid figures are the unions of plane regions which do not lie in the same plane. Matching pictures of solids with the appropriate models should prove useful in helping the children to visualize drawings of 3-dimensional solids. Most pupils seem to be interested in finding objects at home which qualify as cylinders and rectangular boxes and so on.

Solid figures may be identified as blocks, boxes, or balls. For example, a triangular pyramid may be referred to as a block with triangular faces; but it would not be appropriate to identify a ball as a circle or a rectangular prism as a rectangle. Basic distinctions to be made for the children at this time are:

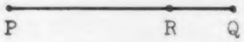
straight edge vs. rounded edge;
flat region vs. rounded region;
flat figure vs. solid figure.

We have stated that in the study of geometry, each of the following objects, a point, a line, and a plane, may be regarded as a primitive element. By these, we can define other geometric objects. Likewise, a 3-space may serve as a primitive element, and it is from this standpoint that we consider points, lines, planes, spaces, as elements of geometry.

### Exercises - Chapter 5

1. Draw a representation of a geometric solid shaped like the pyramids of Egypt. How many vertices does it have?
2. If the union of two rays, $\overrightarrow{AB}$ and $\overrightarrow{CD}$, is a line what will the intersection of $\overrightarrow{AB}$ and $\overrightarrow{CD}$ be?
3. Explain the differences between $\overline{AB}$, $\overrightarrow{AB}$ and $\overleftrightarrow{AB}$.
4. How many different lines may contain:
    a. one certain point?
    b. one certain pair of points?
5. If any of the following statements are false, rewrite them correctly.
    a. A ray has no endpoint.
    b. Two points determine a line segment.
    c. Three points determine a plane.
    d. The intersection of two planes may be a line.

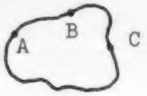
6. Show why $\overline{PQ}$ cannot be divided into disjoint segments so that the union is $\overline{PQ}$.



7. If P and Q are two points of a triangle, what are the possible intersections of $\overline{PQ}$ with the triangle?

8. a. In what ways can a subset of a segment separate the rest of the segment?
   b. Show how some subsets of a segments might not necessarily separate the rest of the segment.

9. a. In the following figure, which of the points, A, B, C, is between the other two?



   b. If three points are connected by a curve, is one point necessarily between the other two?

10. a. If a railroad does not have spur tracks and does not cross itself, what points form the boundary restricting the extent of a train's journey?
    b. To restrict the extent of a ship's operation, what kind of boundary might be required?
    c. What kind of boundary might be required to restrict the extent of an airplane operation?
    d. What kind of boundary might be required to restrict the extent of a submarine operation?

11. Which of the following specifies an interior of a simple closed surface?
    a. A doughnut hole.
    b. A doughnut.
    c. A pat of butter.
    d. The top surface of a kitchen table.
    e. The shell of a coconut.

12. a. How is a pressure cooker changed from a simple closed surface to one that is not?

    b. Does this surface become not closed or not simple?

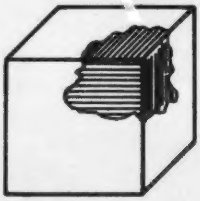
13. Which of the following silhouettes enclose convex sets?
    a. A heart
    b. A kidney
    c. A diamond
    d. An oval
    e. Your profile
    f. An inner tube
    g. A golf ball
    h. A five point star

14. a. Explain why a segment is a convex set.
    b. Is a segment without one endpoint a convex set? without both endpoints?

15. Show how an equilateral triangular region may be "cut" into six congruent regions, into four.

16. If one number is greater than another; what do you know about their positions on the number line?

17. a. If a simple closed curve is drawn on the surface of a cube does it separate the rest of the surface into two disjoint subsets of the surface?
    b. Does it matter which subset is considered the interior?
    c. Describe some simple closed curves that might "cut" the cube into two congruent regions?



18. a. According to the outlines for Books K-3 in Chapter 1, Unit A, in which grades are closed curves presented formally as a topic?

    b. In which grades are topics discussed using basic concepts of closed curves?

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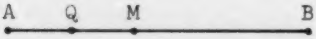
Solutions for Problems

1. 7

2. a. only. The rounded surface intersects the two faces in two edges; b. and d. have both vertices and faces; c. has neither; e. has a vertex as well as a face.

3. (b) and (d) represent curves from A to B; (a) represents a curve from A to A or from B to B. There are two curves represented from A to B, however; (c) is not a curve from A to B because it includes a filled area in its set of points.

4.


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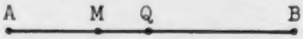
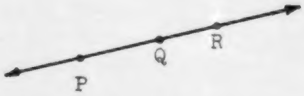
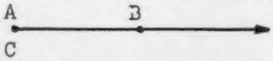


5. Figure corresponds to the first possibility shown in 4. Q is between A and B: M is between A and B; M is between Q and B.

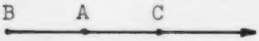
6.


$\overrightarrow{PQ}$ and $\overrightarrow{PR}$ are the same ray or $\overrightarrow{PQ} = \overrightarrow{PR}$

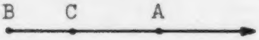
$\overrightarrow{RP}$ and $\overrightarrow{RQ}$ denote the same ray or $\overrightarrow{RP} = \overrightarrow{RQ}$

7. a. $\overrightarrow{AB} = \overrightarrow{CB}$ implies $A = C$. A and C must name the same point, the endpoint of the ray.


b. $\overrightarrow{BA} = \overrightarrow{BC}$ indicates only that A and C are on the same ray. It is not necessary that $A = C$.



or

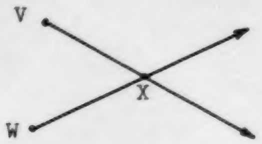


8. a. $\overline{BE}$
   b. B
   c. $\overrightarrow{CA}$
   d. $\overline{CD}$
   e. $\overleftrightarrow{AB}$ or any other notation for the line

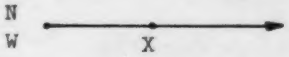
9. a. $\angle PQR$; $\angle Q$; $\angle RQP$
   b. $\overrightarrow{QP}$ and $\overrightarrow{QR}$

10. $\angle CFE$; $\angle CFE$; $\angle CFA$; $\angle BFD$

11. No. There are two possible figures named $\overrightarrow{VX}$ and $\overrightarrow{WX}$:



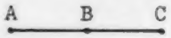
Case 1



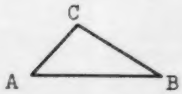
Case 2

In order for $\overrightarrow{VX}$ to be a side of an angle, V must be the vertex, similarly, for $\overrightarrow{WX}$ to be a side, W must be the vertex. This is not true in Case 1. In Case 2, V = W but $\overrightarrow{VX}$ and $\overrightarrow{WX}$ are not two rays, so the definition of an angle is not satisfied.

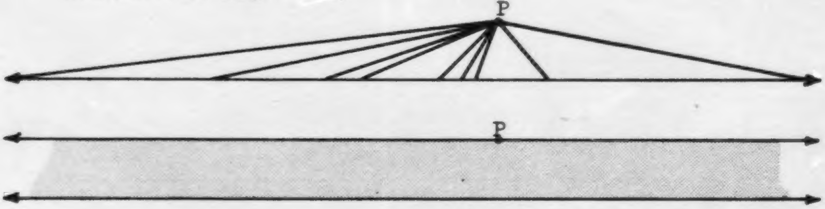
12. a. If one point is between the other two, a line segment is generated.



Otherwise, the figure is a path from A to A which is the union of $\overline{AB}$, $\overline{BC}$ and $\overline{AC}$. It is probably recognized as a triangle.



b. The figure is a band, a plane surface bounded by two non-intersecting lines.

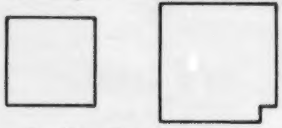


13. The intersection of a line and a plane can be empty, it can contain 1 point or it can be the line. Thus, if two points in a line are in the plane, then all the points in the line must be also in the plane.

14. (b) and (d); The curves (a) and (c) cross themselves once and so are not simple.

15. (b) and (c); (b) is the only curve both simple and closed.

16. (c) The two simple closed curves whose union is figure (c) are



17. D, O are simple closed curves.

C, G, I, J, L, M, N, S, U, V, W, Z are simple but not closed.

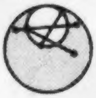
18. a. The curve is simple and closed; therefore the plane is separated into 3 disjoint sets; the interior of the curve, the curve, and the exterior of the curve. E and B are in the interior; D is in the curve; A and C are in the exterior.

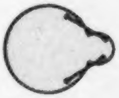
Thus no curve can connect A to B, A to D, or A to E without crossing the curve. A and C can be connected by such a curve, however.

b. D. any curve that contains D intersects the curve at least once, namely at D.

19. (a), (b), and (c).
(d) is not convex although it is a simple closed curve. It is possible to draw a line segment with the tips of the crescent as endpoints. The points between would not lie in the interior of the crescent.
e. and f. are not simple closed curves.

20. (a), (b), (e), (f)
(c) is not a convex set because segments such as those drawn are not entirely within the set.
(d) is not; see figure





21. The triangle; any regular polygon of a specified number of sides.

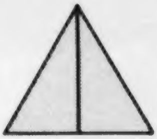
22. A square has four congruent angles and four congruent sides.
A rhombus has four congruent sides.

23. Equilateral triangle.

24. Circular regions. (Note: "circles" is not a correct answer.)

25. You can call it the middle, but the geometric term is midpoint; the midpoint is the intersection of the two congruent segments.

26.



27. 0

28. The coordinate of every point to the right of a given point is greater than the coordinate of the given point.

## ANSWERS TO EXERCISES

### Chapter 1

1. a. {Wednesday}
   b. {pitcher, catcher, first base, second base, shortstop, third base, left field, center field, right field}
   c. {March, April, May, June, July}
   d. { } or ∅
   e. Answers will vary. Example {5, 6, 7}
   f. {Tokyo, London}
   g. {red, orange, yellow, green, blue, indigo, violet}

2. a. {combinations of letters which are pronounced phonetically as an *ef*}
   b. {things that little boys are made of}

3. The intention is that the objects are related only in that they are members of the same set, namely {★, pencil, red}. Someone may be able to think of another common property, however.

4. a. Not equal. 17 and 71 are names for different numbers.
   b. Equal. The sets are the same. Order makes no difference, even though it would be more natural to write {b, u, n, d, l, e}.
   c. Equal. The same elements are listed; order is irrelevant.
   d. Not equal. {zero} has a single member, as opposed to { } which has none. {peacocks native to the North Pole} is the empty set.
   e. Not equal. The members are different.
   f. Not equal. These two sets each have single members but they are not the same member. "Are" and "era" are not names for the same thing.
   g. Equal. Remember. Elements are not listed more than once in a set.

5. a. False. 3 is a *member* of {1, 2, 3}. The braces must be used to indicate set. A correct statement would be: "{3} is a subset of {1, 2, 3}".
   b. True. {ego} is a *subset* because all of its members are also members of {ego, je, I}. It is *proper* because it is not equal to {ego, je, I}.

c. True. Any set equals itself, and every set is a subset of itself.

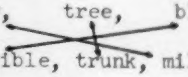
d. False. There are birds that are not hens, a rooster to name one. It would be correct to say {all hens in the world} is a subset of {all birds in the world}.

6. a. {rose, tulip, dandelion} or any subset of this.
   b. {bee, beetle} or any of its subsets.
   c. { }. Some may consider {beetle} but the spelling is different.

## Chapter 2

1. a. A has fewer members than B
   b. C = { cow, tree, blimp}

   D = {dirigible, trunk, milk}



   Answers may vary.

2. Z, X, Y in increasing order.
   Y, X, Z in decreasing order.

3. Mary is taller than Andrea.

4. Q has more members than P.

5. a. and c. only.

   a. To show the 1-1 correspondence, the natural pairing would associate each person and his brain. The question of what is meant by "functioning" brains may enter into consideration. Depending on the answer to this, it may be that there is no 1-1 correspondence.

   b. This is not necessarily one-to-one. Some people have social security numbers and don't file income tax returns. There are many joint income tax returns filed. Corporations file returns but have no social security number. It is conceivable that these two sets might be equivalent, although the natural pairing would not show it.

   c. E is the empty set and so is F. Therefore, E and F are equal. Every set matches itself.

   d. G = {seeing, hearing, tasting, smelling, touching}.
   H = {5}. These sets are obviously not in 1-1 correspondence.

6. A is equivalent to B.
   The members of A and B are in 1-1 correspondence. The members of A can be paired with those of B with none left over.

## Chapter 3

1. a. 4 b. 1 c. 1
   d. 2; the set is $\{d, e\}$.
   e. 1; {the vowels in "bureau"} = $\{u, e, a\}$. However, this set is {the number of vowels in "bureau"} which is the single member set $\{3\}$ whose number property is 1.
   f. 0; there are no counting numbers less than one, so this is the empty set.
2. $N(A) = 4$ $N(B) = 3$ $N(C) = 0$ $N(D) = 3$
   The relationships are: $3 = 3$ $0 = 0$ $4 = 4$
   $4 > 3$ $3 < 4$
   $4 > 0$ $0 < 4$
   $3 > 0$ $0 < 3$
   Some statements can be combined to form two others:
   $0 < 3 < 4$ and $4 > 3 > 0$.
3. Take a set of wide and narrow objects of the same kind. Put the wide ones in a set by themselves and reject the narrow ones. Repeat until the idea gets across, refining distinctions to indicate the relativity of "wide" and "narrow" to some standard. This is an instance of specifying sets according to the property you wish to convey.
4. Answers may vary. The elements may be ordered alphabetically; they may be ordered according to the evolutionary development of man: {amoeba, fish, lizard, ape, man}.
5. $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$. The number is 11. It is now written as a standard subset of the counting numbers whose number is determined by the last member of the set.
6. a. Finite; 10
   b. Infinite
   c. Infinite
   d. Finite; 15. (Recall that "natural numbers" is another name for "counting numbers".)

7. a. Ordinal; one chapter, not the quantity of three chapters, is referred to, namely, the third chapter.
   b. Cardinal; 50 states are mentioned, not the one which is fiftieth.
   c. Ordinal; 1066 suggests one year, not one thousand sixty-six years. It happens to be the one-thousand sixty-sixth year in the set of years A. D.

## Chapter 4

1. a. $A \cup B = \{1, 2, 3, 4, 5\}$
   $A \cap B = \{1, 3, 5\}$
   b. If $B$ is a subset of $A$, then $A \cup B = A$
   c. If $B$ is a subset of $A$, then $A \cap B = B$

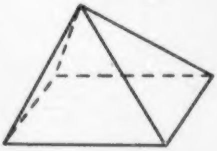
2. The empty set is a subset of any set so if $A$ is a set, $A \cup \{\ \} = A$ and $A \cap \{\ \} = \{\ \}$

3. The cake-mixing operation is not commutative.

4. a. Associative.
   b. Not associative; although the result may be that all items would be consumed, it is likely that mixing mustard with coffee may result in abandoning the meal.
   c. Associative.
   d. Not associative; putting fire with water first, the final mixture will not ignite.

5. The intersection consists of common elements of both sets; the union contains all members of both sets.

## Chapter 5



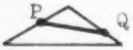
1. Five vertices.
2. If $A = C$, then $\overrightarrow{AB} \cup \overrightarrow{CD}$ is a point; if $A \neq C$, then the union is a segment.
3. $\overline{AB}$ is the segment with $A$ and $B$ as endpoints. $\overrightarrow{AB}$ is the ray with $A$ as endpoint and $B$ a point in the ray, $\overleftrightarrow{AB}$ is the line containing the points $A$ and $B$.

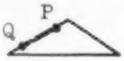
4. Infinitely many different lines may contain a certain point; only one line contains a certain pair of points.

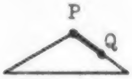
5. a. A ray has one endpoint. b., c., and d. are true.

6. Any segment contains the endpoints. If $\overline{PQ}$ is divided at say, R, the division point either belongs to the segment containing P or to the segment containing Q. If R belongs to both, then $\overline{PR} \cup \overline{RQ} = \overline{PQ}$ but $\overline{PR}$ and $\overline{RQ}$ are not disjoint. If R belongs to $\overline{PR}$, then the set of points in $\overline{RQ}$ without R is not a segment. Crucial to the argument is that for points in a line, there is no very next point; so if R belongs to $\overline{PR}$, there is no next point S to specify $\overline{SQ}$ so that $\overline{PR} \cup \overline{SQ} = \overline{PQ}$.

7.

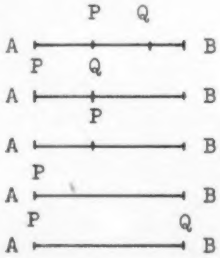






If P and Q are on different sides of the triangle, the intersection consists of just the two points, P and Q. If P and Q are on the same side of the triangle, the intersection is a segment (a subset of the side); this includes the case where P and Q are both vertices of the triangle.

8. a. If $\overline{PQ}$ is a subset of $\overline{AB}$, it may separate the rest of $\overline{AB}$ into two disjoint sets. If $\overline{PQ}$ contains one of the endpoints of $\overline{AB}$, it does not separate the rest of $\overline{AB}$. The subset of $\overline{AB}$ may be just the point P; then again it might or might not separate the rest of $\overline{AB}$.



The subset of $\overline{AB}$ may be $\overline{AB}$; then, there is no separation. The subset may be the empty set; again, there is no separation. If the subset is the union of disjoint segments or points, corresponding analysis may show various kinds of separations.

b. The empty set, the improper subset, a connected segment, or the union of two disjoint segments each containing one of the endpoints would not separate the rest of the segments.

9. a. In this closed curve, each point is between the other two; there is no one point that is between the others.

b. No, as noted in a. above.

10. a. The initial point and the terminal point.

b. A closed curve, as for example, formed by the shore line and a ring of blockading armada.

c. A closed surface, as for example, determined by effective range of antiaircraft defense network.

d. A closed surface with the sea surface as one of the boundaries.

11. b., c., and e. The shell of a coconut specifies the boundary; in so specifying, it also specifies the interior of the boundary.

12. a. When steam escapes from the cooker, releasing the valve is equivalent to poking a hole in the surface. If there is no valve, with enough pressure, the resulting explosion changes the surface to a million and one fragments.

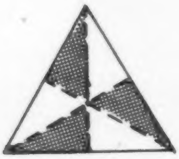
b. The release of the valve changes the surface to one that is not closed.

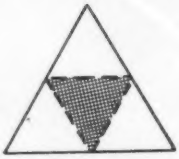
13. c. and d. (g. has dimples.)

14. a. The segment connecting any two points of a given segment lies entirely in the given segment.

b. Yes for both the segment with one or both endpoints deleted.

15.



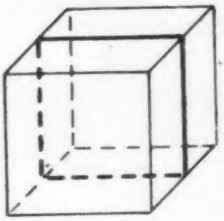


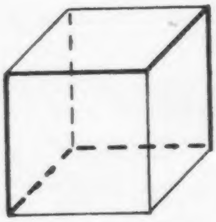
16. The point associated with the larger number is to the right.

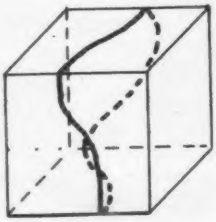
17. a. Yes; as indicated in the drawing.

b. No; it does not matter.

c. For example,







18. a. Grades 1, 2, 3. Closed curves is a section title in Book 1 (Chapter V-2), Book 2 (Chapter III-4), and Book 3 (Chapter I-3).

b. Curves as a basic concept for closed curves is a topic in Book K (Chapter II). Topics using the basic concepts of closed curves are, for example, polygons, Book 1 (Chapter V-3); triangles, Book 2 (Chapter III-5); Book 3 (Chapter I-3, I-5, III-4).